

Markowitz portfolio optimisation – Solver

To use Solver to solve the quadratic program associated with tracing out the efficient frontier (unconstrained efficient frontier UEF) in Markowitz portfolio optimisation it is much more convenient to make use of the COVAR (covariance) function in Excel.

In terms of covariance the Markowitz portfolio optimisation problem is:

$$\text{minimise} \quad \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij} \quad (1)$$

subject to

$$\sum_{i=1}^N w_i \mu_i = R \quad (2)$$

$$\sum_{i=1}^N w_i = 1 \quad (3)$$

$$0 \leq w_i \leq 1 \quad i=1, \dots, N \quad (4)$$

Equation (1) minimises the total variance (risk) associated with the portfolio (where σ_{ij} is the covariance between the returns associated with assets i and j) whilst equation (2) ensures that the portfolio has an expected return of R . Equation (3) ensures that the proportions add to one.

In the Excel sheet seen below we make use of an array formula using MMULT. To see the logic behind this we need to look at the Markowitz objective function in terms of matrix arithmetic.

For simplicity suppose $N=2$ so we have

$$\sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$$

=

$$w_1 w_1 \sigma_{11} + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2 w_2 \sigma_{22}$$

Let us consider the matrix multiplication

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \times \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

so we have the row matrix of weights multiplied by the square covariance matrix multiplied by the column matrix of weights

If we perform the first multiplication

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \times \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

we will get a row matrix (vector) with two elements. The first element will be $w_1\sigma_{11}+w_2\sigma_{21}$ and the second element will be $w_1\sigma_{12}+w_2\sigma_{22}$. This row matrix is

$$\begin{bmatrix} w_1\sigma_{11}+w_2\sigma_{21} & w_1\sigma_{12}+w_2\sigma_{22} \end{bmatrix}$$

Our complete matrix multiplication

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \times \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

therefore becomes

$$\begin{bmatrix} w_1\sigma_{11}+w_2\sigma_{21} & w_1\sigma_{12}+w_2\sigma_{22} \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

=

$$w_1\sigma_{11}w_1 + w_2\sigma_{21}w_1 + w_1\sigma_{12}w_2 + w_2\sigma_{22}w_2$$

The expression we had before from the direct expansion of the summation was

$$w_1w_1\sigma_{11} + w_1w_2\sigma_{12} + w_2w_1\sigma_{21} + w_2w_2\sigma_{22}$$

and these two are the same.

The MMULT term in the Excel sheet below performs the first multiplication, of the row matrix (array) of weights by the square covariance matrix.

In more detail in the Excel sheet shown the MMULT term is in the working cells A14 to E14 and is

$$=MMULT(A12:E12,H12:L16)$$

The risk is shown in cell B17 and is the second matrix multiplication. Here we can make a short cut and just use SUMPRODUCT and so the term in cell B17 is

$$=SUMPRODUCT(A12:E12,A14:E14)$$

so we are taking the SUMPRODUCT of the weights A12:E12 with the working cells A14:E14.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Stock prices							Returns					
2	A	B	C	D	E	Period		A	B	C	D	E	Period
3	916	630.5	440	642	740	0		1.73	1.42	0.11	0.00	2.01	0
4	932	639.5	440.5	642	755	1		-2.33	0.78	0.57	-1.25	0.79	2
5	910.5	644.5	443	634	761	2		-4.32	-2.83	0.67	1.33	-6.66	3
6	872	626.5	446	642.5	712	3		0.23	1.66	4.17	-3.97	-5.34	T=4
7	874	637	465	617.5	675	T=4		-1.17	0.26	1.38	-0.97	-2.30	
8							Average						
9													
10	Weights w()							Covariance COVAR (2dp)					
11	A	B	C	D	E			A	B	C	D	E	
12	0.00	0.00	1.00	0.00	0.00			5.41	3.61	0.85	-2.08	4.59	A
13	Working							3.61	3.28	1.05	-2.59	3.95	B
14	0.85	1.05	2.64	-2.75	-3.34			0.85	1.05	2.64	-2.75	-3.34	C
15								-2.08	-2.59	-2.75	3.83	0.60	D
16	sum w()							4.59	3.95	-3.34	0.60	14.07	E
17	risk												
18	return		desired					do 5 intermediate	Risk	Return	min	max	
19		actual	1.38					points	14.07	-2.30	-2.30	1.38	
20								return step	4.98	-1.69			
21	For an array formula:							0.61	1.24	-1.07			
22	highlight cells where the results will go								0.34	-0.46			
23	type formula								0.02	0.15			
24	do ctrl-shift-enter								0.64	0.77			
25									2.64	1.38			
26													
27													
28													
29													
30													
31													
32													
33													
34													
35													
36													
37													
38													
39													
40													
41													

Solver Parameters

Set Target Cell:

Equal To: Max Min Value of:

By Changing Cells:

Subject to the Constraints:

-
-
-

Capital market line

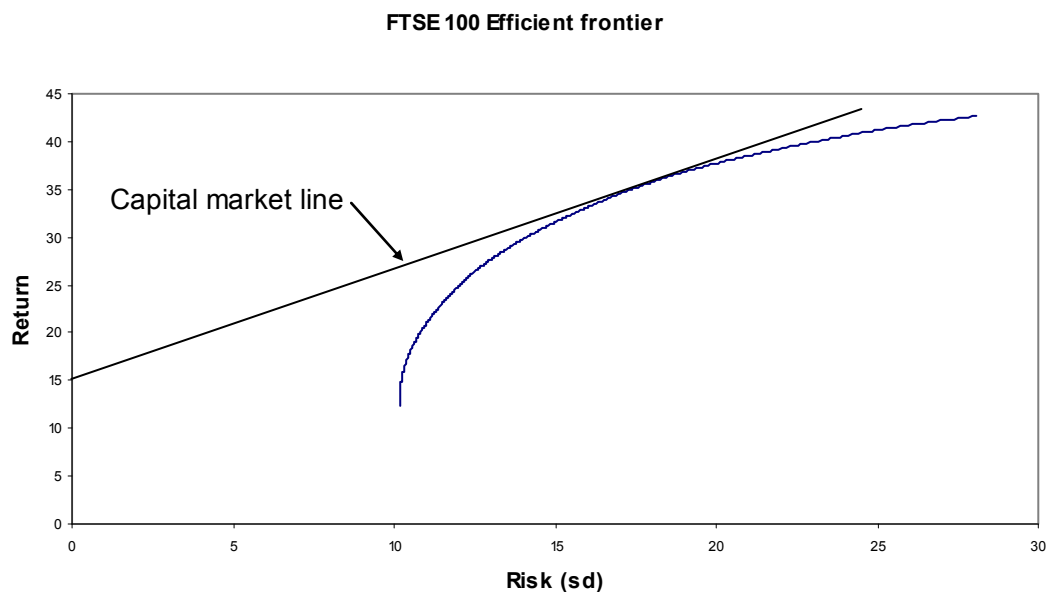
The curve we get when we repeatedly use Solver to find the minimum risk portfolio associated with a given return is a frontier. Parts of that frontier (the portfolios that are efficient, non-dominated) are of interest to us and other parts (in the absence of other considerations) not of interest.

Given a number of portfolios that we have plotted on the efficient frontier how can we choose a single portfolio in which to invest?

One approach is simply to use our human intuition – look at the curve, and make some sort of implicit tradeoff of risk against return.

Another approach is to plot what is known as the **capital market line**. The capital market line is the tangent to the efficient frontier that passes through the risk-free rate on the return axis. This is illustrated below for an example efficient frontier associated with the FTSE100 (assuming a 15% risk-free rate for illustrative purposes).

Note here that when we plot the capital market line we need to express risk on the horizontal axis by standard deviation (not variance).



The portfolio associated with the point at which the capital market line and the efficient frontier meet is the one to choose.

Simply put this portfolio maximises the (linear) tradeoff between risk and return over and above the risk-free rate.

Constraining asset investment

Practical portfolio optimisation inevitably requires that we move beyond the simple Markowitz model seen above and introduce extra constraints into the problem – to better reflect our view as to what we consider an acceptable portfolio. We consider a number of such constraints below.

Maximum proportion

In our Markowitz portfolio optimisation problem we have no constraint on the proportion of the total investment made in each asset (i.e. each w_i can take any value between zero and one).

In practice a large w_i may (even though the portfolio is on the efficient frontier) expose us to an unacceptable degree of risk via putting too much of our total investment into a single asset.

To extend our Markowitz portfolio optimisation problem to the case where we have an upper limit on the proportion of the total investment that can be made in each asset let: δ_i be the maximum proportion that can be invested in asset i

then we have that the problem becomes

$$\text{minimise} \quad \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$$

subject to

$$\begin{aligned} \sum_{i=1}^N w_i \mu_i &= R \\ \sum_{i=1}^N w_i &= 1 \\ 0 \leq w_i &\leq \delta_i \quad i=1, \dots, N \end{aligned}$$

where the only change from before is that whereas previously we had $0 \leq w_i \leq 1$ now we have $0 \leq w_i \leq \delta_i$

The Excel sheet below shows the Solver model with this change implemented with $\delta_i=0.4$

Question – as you make this change would you expect the risk you incur (for a given return) to be more, or less than before? To see what happens and gain insight then:

- find the minimum risk associated with a return of 1.38 (the maximum possible) with $\delta_i=0.4$
- find the minimum risk associated with a return of -1.07 with $\delta_i=0.4$

These cases illustrate that:

- certain returns that previously were achievable become unachievable as you add constraints
- as you add constraints risk (as measured by the Markowitz objective) can only increase

The reason we add constraints is that there are considerations outside the Markowitz objective, we are shaping our portfolio through legitimate considerations of our own by adding constraints to the Markowitz model.

	A	B	C	D	E	F	G	H	I	J	K	L	M	
1	Stock prices							Returns	A	B	C	D	E	Period
2	A	B	C	D	E	Period								0
3	916	630.5	440	642	740	0		1.73	1.42	0.11	0.00	2.01	2.01	1
4	932	639.5	440.5	642	755	1		-2.33	0.78	0.57	-1.25	0.79	0.79	2
5	910.5	644.5	443	634	761	2		-4.32	-2.83	0.67	1.33	-6.66	-6.66	3
6	872	626.5	446	642.5	712	3		0.23	1.66	4.17	-3.97	-5.34	-5.34	T=4
7	874	637	465	617.5	675	T=4	Average	-1.17	0.26	1.38	-0.97	-2.30	-2.30	
8														
9														
10	Weights w(i)							Covariance COVAR (2dp)						
11	A	B	C	D	E			A	B	C	D	E		
12	0.00	0.18	0.40	0.40	0.02			5.41	3.61	0.85	-2.08	4.59 A		
13	upper limit							3.61	3.28	1.05	-2.59	3.95 B		
14	0.40	0.40	0.40	0.40	0.40			0.85	1.05	2.64	-2.75	-3.34 C		
15	Working							-2.08	-2.59	-2.75	3.83	0.60 D		
16	0.26	0.06	0.06	-0.01	-0.07			4.59	3.95	-3.34	0.60	14.07 E		
17														
18	sum w(i)	1												
19	risk	0.03												
20	return	actual	desired											
21		0.15	-1.07											
22														
23	For an array formula:													
24	highlight cells where the results will go													
25	type formula													
26	do ctrl-shift-enter													
27														
28														
29														
30														
31														
32														
33														
34														
35														
36														
37														
38														
39														
40														
41														

Solver Parameters

Set Target Cell:

Equal To: Max Min Value of:

By Changing Cells:

Subject to the Constraints:

\$A\$12:\$E\$12 <= \$A\$14:\$E\$14

\$B\$18 = 1

\$B\$21 = \$C\$21

Sector constraints

An extension to constraining the total investment in any particular asset is to deal with **sector constraints**. Typically this assumes that the assets can be classified as belonging to one of a number of sectors (e.g. energy, banking, telecommunications, etc) and then constraining the total investment in any sector.

For example suppose for the Excel example shown above we have 2 sectors with sector one containing assets {A, B and E} and sector two containing assets {C and D}. Then constraining the amount invested in each sector to be no more than 60% of the total investment (for example) would mean we need the two constraints:

$$\begin{aligned}w_1 + w_2 + w_5 &\leq 0.60 \\w_3 + w_4 &\leq 0.60\end{aligned}$$

Question – here we have shown the two sectors as mutually exclusive, so no asset is in both sectors. In reality do you think you will ever encounter an asset that is in two (or more) sectors?

Note here that for assets in the same sector we might expect to see positive correlation (i.e. they move together, up or down, with the sector). Hence we have, in an implicit way, already included sector constraints (do not make too much investment in the same sector) in our risk objective. But in reality we might do better to include such constraints explicitly, as we have done here.

Trade constraints

So far we have talked purely in terms of the proportion invested in each asset. Indeed you may have had the impression of a pile of cash waiting to be invested. However a moment's thought will reveal that in virtually all practical situations we will have already invested in some existing portfolio of assets. We, when we come to portfolio optimisation, are seeking to change that existing portfolio – perhaps because we feel we can get better performance from a new portfolio. Technically changing an existing portfolio to a new portfolio is known as **rebalancing** the portfolio (or just rebalancing).

In practical situations we also need to consider:

- investing new cash in our portfolio (e.g. new pension contributions we might have received); or
- taking cash out of the portfolio (e.g. to meet liabilities, necessitating selling some of the assets in our existing portfolio)

However to ease the mathematical discussion here we will neglect these issues.

Suppose X_i is the number of units of asset i that we currently own. As a result of portfolio rebalancing we end up with x_i units of asset i (where x_i is a variable that will be decided as a result of the optimisation). Suppose, for the sake of illustration, we wish to restrict the amount of trading in asset i that we do to T_i units – so we do not wish to trade more than T_i units of asset i , where T_i is our choice – not something the

optimisation can decide for us. Then the trade constrained portfolio rebalancing problem, assuming we have zero transaction cost, is:

$$\text{minimise} \quad \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$$

subject to

$$\sum_{i=1}^N w_i \mu_i = R$$

$$\sum_{i=1}^N w_i = 1$$

$$0 \leq w_i \leq 1 \quad i=1, \dots, N$$

$$X_i - T_i \leq x_i \leq X_i + T_i \quad i=1, \dots, N$$

$$w_i = (x_i \times \text{current price of asset } i) / \sum_{i=1}^N (X_i \times \text{current price of asset } i) \quad i=1, \dots, N$$

where the first set of equations are as before.

The equation $X_i - T_i \leq x_i \leq X_i + T_i$ ensures that the number of units of asset i after rebalancing (x_i) lies within the desired amount T_i of the number of units X_i before rebalancing.

The last equation here defines the proportion (w_i) invested in each asset in terms of the number of units (x_i) of that asset held.

Minimum proportion

Sometimes we may end up with a very small proportion w_i value as a result of the optimisation – indicating that we should just invest a small amount in asset i . This may be administratively inconvenient and so we might wish to impose a constraint upon the minimum amount invested in each asset, say 1% of our total investment. This can be easily done via the constraint $w_i \geq 0.01 \quad i=1, \dots, N$.

Try this for yourself in one of the Solver models – how many assets do you end up with in your portfolio? Is this what you expected to happen?

This example illustrates that in reality we wanted the constraint “*if we invest, and we may not, then our investment must have $w_i \geq 0.01$* ”. In other words we wanted the constraint “*either $w_i = 0$ or $w_i \geq 0.01$* ”. This latter constraint cannot be achieved unless

we expand our model to include integer variables, which we will do as we turn to cardinality constrained portfolio optimisation.