CHAPTER 1

Markowitz model

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1. Notations and model setup

We specify the model. In this section, there are two dates $t = 0$ and $t = 1$ with trading possible at time $t = 0$. At $t = 0$ prices of all assets are known and exogenous given. Therefore, we are in the setup of partial equilibrium where agents have no influence on prices (they are so-called price takers).
The sample space Ω and its elements ω can either be a finite set or consist of a continuum of elements ω. The elements ω are called *states of the world*. The value of each state is assumed to be unknown to the investors but will be apparent at $t = 1$ to the them. On the sample space a probability P is defined and we consider a standard probability space $(\Omega, \mathcal{F}, P)$.
The following financial instruments exist. A *cash - or bank account* B_t where the price at $t = 0$ is normalized to 1 ($B_0 = 1$). The price at $t = 1$ of the bank account can be stochastic or deterministic, in any case it is larger than 1 ($B_1(\omega) \geq 1$ for all ω). If the price at $t = 1$ is deterministic, we call B_t the *risk-free asset*. The *interest rate* R is defined by

$$R = \frac{B_1 - B_0}{B_0} = B_1 - 1 \geq 0 \ ,$$

which can be stochastic or deterministic according to the bank account under consideration. Further examples of riskless assets are government bonds or bonds issued by firm with a maximum rating. Although strictly speaking this instruments are not risk free, since the default risk is for example not zero, this construction is at least a good approximation to some real existing instruments.

The risky assets $S_j(t), j = 1, \ldots, N$ are defined as follows: Their prices at $t = 0$ are known and positive scalars. Their prices at $t = 1$ are non-negative random variables whose value become known to the investors only at time $t = 1$. If there is only a single asset, we write S_t by abuse of notation.
A *trading strategy* $\psi = (\psi_0, \ldots, \psi_N)$ describes the investor's portfolio as carried forward from time $t = 0$ to time $t = 1$. The component ψ_0 is the amount of CHF invested in the savings account and $\psi_j, j > 0$, is the number of units of the security j held between the two times. If the components are negative, we speak about a short position for the risky assets or borrowing for the bank account. We use the word trading strategy and portfolio of an investor as synonyms.
A *primitive financial market* then consists of two dates, a probability space, the securities with the exogenous given prices $B_t, S_1(t) \ldots, S_N(t)$, the trading strategies and the assumption, that all investors prefer more money to less.
In this setup, we introduce various important quantities.
The *value process* V_t^ψ *(due to the strategy* ψ*)* is defined by

$$V_t^\psi = \psi_0 B_t + \sum_{j=1}^N \psi_j S_j(t) \ .$$

The initial value V_0 is the *initial wealth* of the investor. With the *difference operator* Δ, defined by

$$\Delta X_j(t) = X_j(t) - X_j(t-1)$$

the total profit or loss of a strategy ψ is expressed by the *gain process* G_t

$$G^\psi(t) = \psi_0 r + \sum_{j=1}^N \psi_j \Delta S_j(t) \ .$$

It is useful to consider *discounted* values of future prices. The reason of discounting is due to the need of comparing relative future prices of securities; hence we normalize the prices. The discount instrument, which is called the *numeraire*, is in standard theory the bank account (Black-Scholes model, Cox-Ross-Rubinstein model for example). But for more advanced models, choosing the bank account as numeraire is not optimal. If Y_t is used as numeraire for the security X_t, the discounted variable is denoted by

$$\tilde{X}_t = \frac{X_t}{Y_t} \ .$$

Instead of working with prices, we often use *return.* The rate of return $R_j(t)$ of a security j at time t is defined by

$$R_j(1) = \frac{S_j(1) - S_j(0)}{S_j(0)} \ .$$

In our single period models the return of the cash account is the interest rate R. The rate of return of R_j instrument j is a random variable for risky assets and, typically assumed to be a real number for riskless assets.
If R^ψ is the return of a portfolio corresponding to a strategy ψ and if $V_0 > 0$, we have

$$R^\psi = \frac{V_1^\psi - V_0^\psi}{V_0^\psi} \ .$$

Aside the trading strategies or portfolios defined for the absolute security prices it is often more natural to use the *normalized* strategies

$$\phi = (\phi_0, \dots \phi_N) \ , \phi_0 = \frac{\psi_0}{V_0} \ \ \phi_k = \frac{\psi_k S_k(0)}{V_0} \ , \ k = 1, \dots, N \ .$$

Therefore, ϕ_k is the fraction of initial wealth invested in the security k. Since there is a one-to-one relationship between prices and returns, we are free to work with either of the two strategies. The following proposition summarizes the basic relationship of all the expressions defined so far.

Proposition 1 (Bookkeeping Proposition).

$$\begin{aligned}
V_1^\psi &= V_0^\psi + G^\psi \\
\tilde{G} &= \sum_{j=1}^N \psi_j \Delta \tilde{S}_j \\
\tilde{V}_1 &= \tilde{V}_0 + \tilde{G} \\
G &= \psi_0 B_0 R + \sum_{j=1}^N \psi_j S_j(0) R_j \\
\tilde{S}_j(1) - \tilde{S}_j(0) &= S_j(0) \frac{R_j - R}{1 + R} \\
R^\psi &= \phi_0 R + \sum_{j=1}^N \phi_j R_j \ .
\end{aligned}$$

The proof is left to the reader.

Definition 2. *Suppose that there are N risky assets and that a trading strategy ϕ is given. The vector of expected values $E[R^\phi]$ and the covariance matrix are denoted by*

$$\mu \in \mathbf{R}^N \ , \ \mu_j = E[R_j^\phi] \ , \ V_{kl} = \mathit{cov}\,(R_k^\phi, R_l^\phi) \ .$$

The expectations $E[\bullet]$ always is meant with respect to the (objective) probability P.
From the definitions we immediately get

PROPOSITION 3. *The expected rate of return of a portfolio and the variance are given by*

$$E[R^\phi] = \langle \mu, \phi \rangle \ , \ var(R^\phi) = \langle \phi, V\phi \rangle \ .$$

2. Unrestricted Mean-variance analysis

We discuss in this section the classical Markowitz model.

DEFINITION 4. *A portfolio ϕ^* is mean-variance efficient if there exist no portfolio ϕ such that*

$$E[R^\phi] \geq E[R^{\phi^*}] \ , \ var(R^\phi) < var(R^{\phi^*}) \ . \tag{1}$$

Instead of mean-variance efficient portfolios we often simply speak about efficient portfolios. It is important to note what we are actually doing at this point. We introduce a criterion to distinguish so-called efficient portfolios from other ones. So far, this is a purely *ad hoc* criterion and a theoretical basis - for example a decision theoretic foundation - is missing at this point. Simply, we start to discuss portfolio selection by agreeing that the first and second moment of a portfolio solely matter for selection. The question, whether there is any theoretical foundations is discussed in Chapter X. For the rest of this chapter we agree on ad hoc rules and see how far we can go in theory using them and how do they fit with real data.
The criterion is based on a return-risk trade-off. Therefore, the Markowitz theory assumes that an investor not only values return but also risk matters. Hence, preferences are more involved than preferences where we only agree that “more money is better than less”.
Beside mean-variance efficient portfolios, it is often more convenient to work with the larger class of *minimum-variance portfolios*. By definition, this set contains the mean-variance efficient portfolios but it also includes the single portfolio with the smallest variance at every level of expected return. In Figure **??** the two different geometric loci are shown.
Another rule which we state is that for any investor using the mean-variance criterion, the opportunity not to invest all of his wealth is dominated by *any* investment in the financial assets. Therefore, all of his wealth is invested. Finally, there is no possibility of borrowing in this model. In summary the model's assumption are

ASSUMPTION 5 (Classical Markowitz model). *The classical mean-variance model of Markowitz is defined as follows.*

(1) *There are N risky assets and no risk free asset. Prices of all assets are exogenous given. In other words, investors are price takers and they do not affect prices.*
(2) *There is a single time period ($t = 0, t = 1$).*
(3) *There is probability space $(\Omega, \mathcal{F}, P)$.*
(4) *There are no transaction costs.*
(5) *Markets are liquid for all assets.*
(6) *Assets are infinitely divisible (we may buy π units of Roche).*
(7) *Full investment and no borrowing hold, i.e.*[3]

$$\langle e, \phi \rangle = 1$$

with $e = (1, \ldots, 1)' \in \mathbf{R}^n$.
(8) *Portfolios are selected according to the mean-variance criterion.*

[3] The notation $\langle x, y \rangle = \sum_i x_i y_i$ denotes the scalar product.

The assumption that investors are price takers has some far reaching consequences which often are neglected by practitioners. Since price formation in the stock markets for example is a result of individual interactions, the model we consider in this section neglects this by assuming that prices fall from heaven. Although for small investors the price taker behavior is a reasonable approximation, for large investors this is certainly not true. More basic than this distinction according to the wealthiness of the investors, is the question whether the mean-variance criterion in selecting portfolios is "supported by the market". First, it is not at all clear what "supported by the market" means. A rigorous characterization in economics is based on general equilibrium theory: Roughly, we consider an arbitrary number of different individuals each equipped with an endowment, preferences over consumption and an optimization program. The program consists in finding consumption and investment such that objective function (utility function) is optimized for each individual. An equilibrium is then defined by:

(1) Prices for the assets such that the asset markets clear[4],
(2) Investment strategies (portfolios) and consumption plans for all individuals, such that the objective of each individual while respecting the individual budget constraints.

If such a pair of prices and actions (the trading strategies, consumption plans) is found, we call them a *general financial equilibrium*.
Answering this type of questions is much more harder than to solve the problem we face in this section. But considering the equilibrium questions is crucial to understand whether mean-variance analysis decision making can be consistent with equilibrium behavior. Hence, we define mean-variance analysis to be "supported" if individuals choose this criterion in the equilibrium model as their optimal trading strategy. Clearly we hope that there exist models which support mean-variance analysis under very weak assumptions.
As a final comment we note that many bank runs in the last years occurred because the management did not understand what the essential questions are if models are implemented in the banks: First, one should ask how meaningful the model is *per se*. In other words, does the model makes economic sense? Second, since it is only a model, what are the issues not or not completely considered within the model. In other words, only the assumptions the quants[5] consider matter for the outcome of the model. Third, does the bank possess the data with the required quality to determine the parameters of the model? Fourth, and this is the most difficult part, what are the consequences of the bank's decision according to the model if *circumstances* are changing. Circumstances can be regulatory requirements or behavior of competitors for example. Finally, if a model is used to produce benchmarks for asset managers for example, the managers have then an incentive to "beat" the benchmark since their payment depend on how well they do relative to the benchmark. If we assume that the model implemented is ill-behaved, then the manager has an incentive to beat an ill-behaved model. For example, he is encouraged or forced to face extreme risky positions. Hence, models may lead to incentive distortions for the asset managers which can turn out to be disastrous for the bank itself.
After this comments, we return to the mean-variance problem.

The question is, how can we systematically determine the mean-variance portfolios. Intuitively, a goal is to minimize the variance of the portfolios for a given expected return. In fact, that the following optimization is the key to find efficient

[4]This means, again loosely spoken, that demand and supply for the assets equalize.
[5]People in the financial industry with a math background are often called like this.

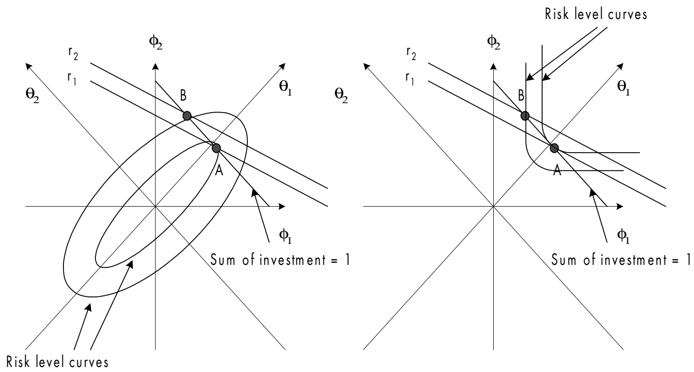

FIGURE 1. Illustration of the technical assumptions 6.

portfolios.

$$\begin{aligned} \min_{\phi} \quad & \frac{1}{2}\langle \phi, V\phi \rangle \qquad (\mathcal{M}) \qquad (2)\\ s.t. \quad & S = \{\phi \in \mathbf{R}^N | \ \langle e, \phi \rangle = 1 \ , \ \langle \mu, \phi \rangle = r\} \ . \end{aligned}$$

(1) The admissible set does not excludes *short selling*, i.e. portfolio positions $\phi_k < 0$.
(2) The parameter r is exogenously given[6].
(3) The Markowitz model $\mathcal{M}$ is a quadratic optimization problem (quadratic objective function and linear constraints). The feasibility set S is convex since it is the intersection of two hyperplanes.
(4) The factor $\frac{1}{2}$ is chosen for notational convenience.
(5) The solution(s) of the program depend(s) on the parameter r.

We impose the following technical conditions.

ASSUMPTION 6.
The covariance matrix is strictly positive definite.
The vectors e, μ are linearly independent.
All first and second moments of the random variables exist.

The positivity of the covariance matrix means that all N assets are indeed risky. This also holds true for convex combinations of these assets. The linear independence condition avoids a degenerate situation where the constraints in the model $\mathcal{M}$ are contradicting unless

$$r = \frac{\langle \mu, e \rangle}{N}$$

holds which is a non-generic situation.
We consider an example to illustrate the technical assumption. Suppose that two assets are given and we assume that the covariance matrix is in the first case strictly positive definite (two strictly positive eigenvalues) and indefinite in the second case (a positive and a negative eigenvalue, see Figure 2). In the figure, risk level curves and the two restrictions are shown. The restrictions are straight lines where we considered for the expected rate of return two case parameterized by the returns $r_1 < r_2$, respectively.

[6]"Exogenous" is used by economists to state that parameters, states or other type of variables in a model are given and not chosen by the decision maker. "Endogenous" means that the corresponding quantity is chosen by the decision maker in the respective model.

In the first case, if we further assume that the covariance matrix is symmetric, the risk level curves are ellipses and hyperbolas in the second case (the original coordinate system as well as the main-axis coordinate system are shown in the figure). In the first case, if a higher return r_2 is the goal of the investor, also the corresponding risk level (=variance) increases. This is illustrated by the two points A and B. In the second case, a transition from A to B contrary leads to a lower risk exposure given a higher expected rate of return. Hence, the investor will always chose zero investment in security ϕ_1 and full investment $\phi_2 = 1$, independent of the risk an return characteristics of the securities. The figure also shows that the linear independence between the two vectors μ and e is necessary, else the two lines do not intersect (unless in the non-generic case) and there does not exist a solution.

PROPOSITION 7. *If the Assumptions 6 hold, then the model $\mathcal{M}$ has a unique solution.*

PROOF. The constraints are linear, hence convex. Since the intersection of convex sets is convex (Proposition X), S is convex. The objective function is strictly convex due to 6. The second condition in 6 implies that the gradients of the constraints are linearly independent, i.e. the Mangasarian-Fromowitz constraint qualification holds which implies that the Abadie CQ holds. Therefore, all conditions of Proposition X are satisfied. □

PROPOSITION 8. *If the Assumptions 6 hold, the solution of the model $\mathcal{M}$ is*

$$\phi^* = r\phi_0^* - \phi_1^* \tag{3}$$

with

$$\begin{aligned} \phi_0^* &= \frac{1}{\Delta}\left(\langle e, V^{-1}e\rangle V^{-1}\mu - \langle e, V^{-1}\mu\rangle V^{-1}e\right) \\ \phi_1^* &= \frac{1}{\Delta}\left(\langle e, V^{-1}\mu\rangle V^{-1}\mu - \langle \mu, V^{-1}\mu\rangle V^{-1}e\right) \\ \Delta &= ||\sigma^{-1}e||^2||\sigma^{-1}\mu||^2 - (\langle\sigma^{-1}e, \sigma^{-1}\mu\rangle)^2 \\ V &= \sigma\sigma' \ , \ ||x|| = \sqrt{\langle x, x\rangle} \ . \end{aligned} \tag{4}$$

PROOF. Forming the Lagrangian, the first order conditions or the KKT conditions are

$$\begin{aligned} 0 &= V\phi - \lambda_1 e - \lambda_2\mu & (5)\\ 1 &= \langle e, \phi\rangle & (6)\\ r &= \langle \mu, \phi\rangle \ . & (7) \end{aligned}$$

Since V is strictly positive definite, V^{-1} exists and from (6) follows

$$\phi = \lambda_1 V^{-1}e + \lambda_2 V^{-1}\mu \ .$$

Multiplying this last equation from the left with e and μ, respectively, and using the normalization condition and the return constraint, we get

$$\begin{aligned} 1 &= \lambda_1\langle e, V^{-1}e\rangle + \lambda_2\langle e, V^{-1}\mu\rangle \\ r &= \lambda_1\langle \mu, V^{-1}e\rangle + \lambda_2\langle \mu, V^{-1}\mu\rangle \ . \end{aligned} \tag{8}$$

If we set $\tau = (\lambda_1, \lambda_2)'$ and $y = (1, r)'$ the last system reads

$$y = \begin{pmatrix} \langle e, V^{-1}e\rangle & \langle e, V^{-1}\mu\rangle \\ \langle \mu, V^{-1}e\rangle & \langle \mu, V^{-1}\mu\rangle \end{pmatrix} \tau =: A\tau \ . \tag{9}$$

The matrix A is invertible, since $\det A = \Delta > 0$. This follows from the linear independence of e and μ which implies that $\sigma^{-1}e$ and $\sigma^{-1}\mu$ are linearly independent and the Cauchy-Schwartz inequality $\langle x, y\rangle^2 \leq ||x||^2||y||^2$ which is a strict inequality if the vectors x and y are linearely independent. Therefore, the system $y = A\tau$ has a unique solution $\tau = A^{-1}y$. Considering the components of this matrix equation and inserting them into $\phi = \lambda_1 V^{-1}e + \lambda_2 V^{-1}\mu$ implies (3). For further reference, we note the optimal multiplier values:

$$\lambda_1 = (A^{-1}y)_1 = \frac{1}{\Delta}\left(-\langle\mu, V^{-1}\mu\rangle + r\langle e, V^{-1}\mu\rangle\right) \quad (10)$$

$$\lambda_2 = (A^{-1}y)_2 = \frac{1}{\Delta}\left(-\langle e, V^{-1}\mu\rangle + r\langle e, V^{-1}e\rangle\right) . \quad (11)$$

□

We define the following expressions, which appear very frequent in the rest of this chapter.

DEFINITION 9.

$$a = \langle\mu, V^{-1}\mu\rangle \ , \ b = \langle e, V^{-1}e\rangle \ , \ c = \langle e, V^{-1}\mu\rangle \ .$$

The following bounds hold.

PROPOSITION 10. *In the model $\mathcal{M}$, the following bounds hold true:*

$$b \in [\frac{N}{\lambda_{max}}, \frac{N}{\lambda_{min}}] \ , \ c \in [\frac{||\mu||^2}{\lambda_{max}}, \frac{||\mu||^2}{\lambda_{min}}] \ , \ |a| \leq \frac{\sqrt{N}||\mu||}{\lambda_{min}} \quad (12)$$

with λ_{max} (λ_{min}) the maximum (minimum) eigenvalue of V.

PROOF. Sieve V is strictly positive definite, $V > 0$, $\langle e, V^{-1}e\rangle > 0$ Follows. For $e = Uy$ with U is a $N \times N$ orthogonal matrix, we get

$$\begin{aligned} b = \langle e, V^{-1}e\rangle &= \langle y, U'V^{-1}Uy\rangle = \sum_i \lambda_i^{(V^{-1})} y_i^2 \\ &\leq \lambda_{max}^{(V^{-1})}\langle y, y\rangle = \lambda_{max}^{(V^{-1})}\langle e, U'Ue\rangle = \lambda_{max}^{(V^{-1})}||e||^2 \\ &= \frac{1}{\lambda_{min}}N, \end{aligned}$$

where $U'V^{-1}U$ is a diagonal matrix. Similar, a lower bound follows for the smallest eigenvalue which proves

$$\langle e, V^{-1}e\rangle \in [\lambda_{min}^{(V^{-1})}||e||^2, \lambda_{max}^{(V^{-1})}||e||^2] = [\frac{N}{\lambda_{max}}, \frac{N}{\lambda_{min}}] \ .$$

The proof of the second claim is analogous. To prove the last inequality, we first note that the matrix

$$A = (e,\mu)'V(e,\mu) = \begin{pmatrix} \langle e, V^{-1}e\rangle & \langle e, V^{-1}\mu\rangle \\ \langle e, V^{-1}\mu\rangle & \langle\mu, V^{-1}\mu\rangle \end{pmatrix}$$

is invertible due to the independence of the vectors e and μ. The Cauchy-Schwarz inequality implies that $\det A > 0$ which is equivalent to

$$|\langle e, V^{-1}\mu\rangle| \leq \sqrt{\langle\mu, V^{-1}\mu\rangle\langle e, V^{-1}e\rangle} \leq \frac{\sqrt{N}||\mu||}{\lambda_{min}} \ .$$

□

In the next step of the analysis, we calculate the corresponding variance for the optimal portfolio ϕ^* of Proposition 8. We get

$$\sigma^2(r) = \langle\phi^*, V, \phi^*\rangle = \langle r_*\phi_0^* - \phi_1^*, V(r_*\phi_0^* - \phi_1^*)\rangle = \frac{1}{\Delta}\left(r^2 b - 2rc + a\right) . \quad (13)$$

The locus of this set in the $(\sigma(r), r)$-space are hyperbolas (see Figure 2). Proposition 3 implies that the function

$$\sigma(r) = \sqrt{\langle\phi^*, V\phi^*\rangle}$$

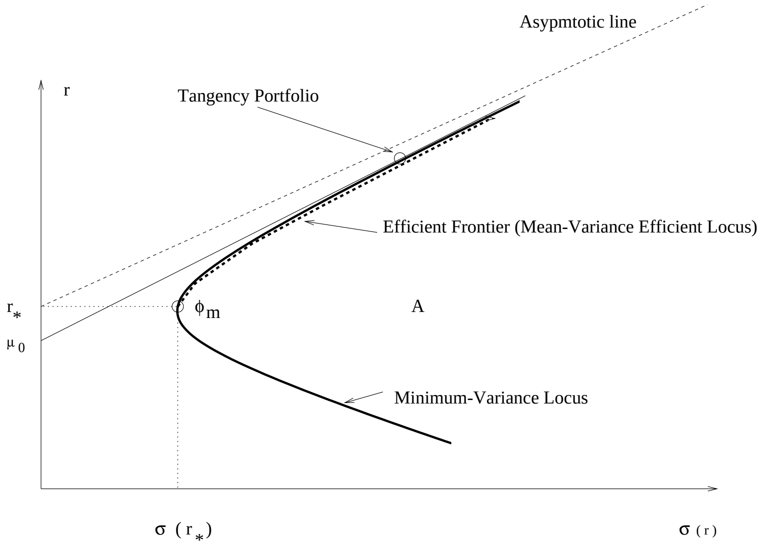

FIGURE 2. Mean-variance illustrations in the $(\sigma(r), r)$-space. The asymptotic line is given by $r = \sigma + r_* \sqrt{\langle \phi_0^*, V\phi_0^* \rangle}$ and $C = c$.

provides the minimum standard deviation and variance for any given mean r. Since V is positive definite, the quadratic form under the root is convex and positive. Because the root is strictly increasing on the positive real numbers from Proposition **??** follows that the function $\sigma(r)$ is convex. Therefore, the function has a unique minimum r_* which is given by the solution of

$$\sigma'(r) = \frac{\langle \phi_0^1, V(r\phi_0^* - \phi_1^*) \rangle}{f(r)} = 0,$$

i.e.

$$r_* = \frac{\langle \phi_0^*, V\phi_1^* \rangle}{\langle \phi_0^*, V\phi_1^* \rangle} = \frac{c}{b} \ . \tag{14}$$

With the results of Proposition 8 we can calculate the *global minimum variance portfolio* $\phi_m^* = \phi_m^*(r_*)$. We note that the components ϕ_0^* and ϕ_1^* in the optimal portfolio (4) are independent on the value r. We then have

$$\begin{aligned} \phi_m^*(r_*) &= r_*\phi_0^* - \phi_1^* \\ &= \frac{c}{b}\frac{1}{\Delta}\left(bV^{-1}\mu - cV^{-1}e\right) - \frac{1}{\Delta}\left(cV^{-1}\mu - aV^{-1}e\right) = \frac{1}{\Delta}(a - \frac{c^2}{b})V^{-1}e \end{aligned} \tag{15}$$

Therefore,

$$\phi_m^*(r_*) = \frac{1}{b}V^{-1}e \tag{16}$$

and the *global minimum variance* for this strategy is

$$\sigma(r_*)^2 = \frac{1}{b} \ . \tag{17}$$

It is interesting to note that the global minimum variance is independent of the return properties of the assets.
We finally define the efficient frontier and the notion of dominance.

DEFINITION 11. *The set*

$$A := \{(E[R^\phi], \sqrt{var(E[R^\phi]\ )}|\ \phi \in \mathbf{R}^N\ , \langle \phi, e \rangle = 1\} \tag{18}$$

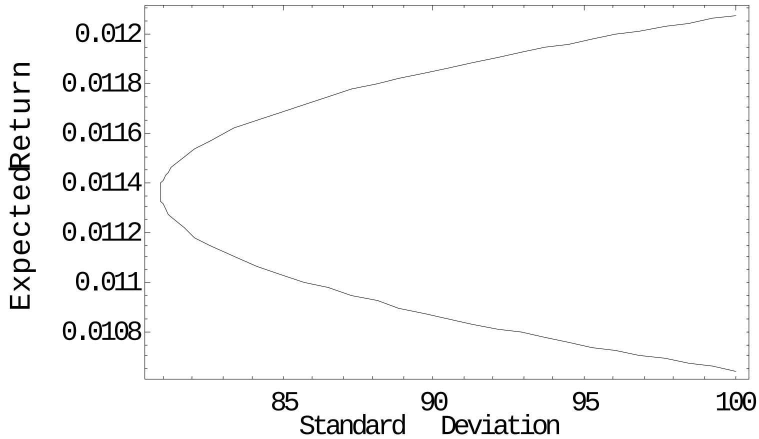

FIGURE 3. The figure shows the minimum variance locus for the example with three risky assets: The Standard& Poor's index 500, US Governement Bonds and a US Small Cap Index.

is the set of mean/standard deviation portfolios and the set

$$\partial_+ A := \{(r, \sigma(r))|\ r \geq r_*\}\ ,\ \partial_- A := \{(r, \sigma(r))|\ r < r_*\} \tag{19}$$

is the efficient (inefficient) frontier (of the Markowitz model). Finally, if two portfolios have the same mean, the one with the lower standard deviation is said to dominate the one with the higher standard deviation.

The following facts are straightforward to verify:

- For all portfolios on the efficient frontier there exists no other portfolio with the same mean and a lower standard deviation. In other words the portfolios on the efficient frontier are not *dominated* by any other portfolio in A.
- For each inefficient portfolio exists an efficient portfolio with the same variance but a higher expected rate of return.
- It is straightforward to prove that any minimum variance portfolio, i.e. it satisfies (3), is an efficient portfolio if

$$r \geq \frac{c}{b} = r^*$$

holds.

We consider the following simple example to illustrate the theory developed so far. We consider the case of three risky instruments: The Standard& Poor's index 500, US Government Bonds and a US Small Cap Index. This example shows that in the mean variance analysis the risky instruments need not be pure assets but they can be indices of several assets. On a monthly basis the expected returns are

$\mu = (0.0101, 0.00435, 0.0137)' =$ (Standard& Poor, US Gov. Bonds, Small Cap)$'$

and the covariance matrix V is

$$V = \begin{pmatrix} 0.003246 & 0.000229 & 0.004203 \\ 0.000229 & 0.000499 & 0.000192 \\ 0.004203 & 0.000192 & 0.007640 \end{pmatrix} .$$

Figure 2 shows the minimum variance locus for this example.

The next proposition characterizes the efficient frontier in terms of the expected returns, variances and covariances of the returns.

PROPOSITION 12. *Assume that the Assumptions 6 and*

$$r \geq \frac{c}{b}$$

hold in the model $\mathcal{M}$. A portfolio ϕ is efficient iff a positive affine relation between the covariance of the return of each asset R_i with the portfolio R^ϕ and the expected return exists. Formally,

$$cov(R_i, R^\phi) = f_1^\phi E[R_i] + f_2^\phi \ , \ f_1^\phi \geq 0 \ , \ i = 1, \ldots, N \ . \tag{20}$$

PROOF. The vector of the covariances of the returns with the portfolio is obtained by calculating $V\phi$. To prove necessity, we assume that the portfolio ϕ is efficient. It follows from (4)

$$V\phi = \frac{rb - c}{\Delta}\mu + \frac{a - rc}{\Delta}e$$

i.e.

$$cov(R_i, R^\phi) \quad = \quad \frac{rb - c}{\Delta}\mu_i + \frac{a - rc}{\Delta} = \frac{bE[R^\phi] - c}{\Delta}E[R_i] + \frac{a - cE[R^\phi]}{\Delta} \ . \tag{21}$$

We have to verify $\frac{bE[R^\phi]-c}{\Delta} \geq 0$. Linear independence of the vectors μ, e and the Cauchy-Schwarz inequality imply $\Delta > 0$. Since $E[R^\phi] \geq \frac{c}{b}$, necessity is proven. For the minimum variance portfolio we note that $E[R^{\phi^*_{min}}] = \frac{c}{b}$. To prove sufficiency, we assume that (20) holds true. In vector notation this reads

$$V\phi = f_1^\phi \mu + f_2^\phi e \ , \ f_i^\phi \geq 0 \ .$$

The weights f_i^ϕ of this portfolio are obtained by multiplication from the left with V^{-1} and using the budget restriction $\langle \mu, \phi \rangle = E[R^\phi]$ and the normalization conation. Solving the two equation with respect to f_1^ϕ, f_2^ϕ and inserting the results into (20) implies that (4) is satisfied with $E[r^\phi] \geq \frac{c}{b}$. □

To interpret the condition (20) we make use of the following result.

PROPOSITION 13. *In the Markowitz model $\mathcal{M}$*

$$\frac{\partial \sigma^2(r)}{\partial \mu_k} = cov(R_k, R^{\phi^*}) \tag{22}$$

holds with ϕ^* a minimum variance portfolio.

PROOF. Exercise. □

It follows from this proposition that the impact of one unit more return in asset k on the optimal variance equals the covariance of asset k with the minimum variance portfolio. If the asset k is positively correlated with the portfolio, a unit more return of this asset increases the variance and the contrary holds, if the correlation is negative. Hence, to reduce the variance as much as possible, a combination of negatively correlated assets should be chosen. This is the so-called *Markowitz phenomenon.* With the result in Proposition 13, the condition (20) can can be interpreted as follows. A portfolio is mean-variance efficient iff the marginal contribution of each security to the portfolio risk is an positive, affine function of the expected return.
The next goal is to rewrite the condition (20) in a form which is most widely used.

PROPOSITION 14. *Assume that the Assumptions 6 and*

$$r \geq \frac{c}{b}$$

hold in the model $\mathcal{M}$. A portfolio ϕ is efficient iff - with expectation of the global minimum variance portfolio - there an **uncorrelated, arbitrary** *portfolio $\bar{\phi}$, such that*

$$E[R_i^{\phi}] = E[R_i^{\bar{\phi}}] + cov(R_i, R^{\phi})\frac{E[R^{\phi}] - E[R^{\bar{\phi}}]}{\sigma^2(R^{\phi})} \ , \ i = 1, \dots, N \tag{23}$$

with $E[R^{\phi}] - E[R^{\bar{\phi}}] > 0$.

Proof. Exercise. □

The interpretation of (35) is the same one than for (20) . The slope of the affine relationship between expected return and the covariance is $\frac{E[R^{\phi}]-E[R^{\bar{\phi}}]}{\sigma^2(R^{\phi})}$. It is widely used to define $\beta_i = \frac{cov(R_i,R^{\phi})}{\sigma^2(R^{\phi})}$ and the vector of betas

$$\beta = (\frac{cov(R_N, R^{\phi})}{\sigma^2(R^{\phi})}, \dots, \frac{cov(R_N, R^{\phi})}{\sigma^2(R^{\phi})})'$$

is then a measure of the covariance of the assets with the efficient portfolio normalized by the variance of the efficient portfolio.
In the model of Markowitz we consider the return restrictions $\langle \mu, \phi \rangle = r$. It is reasonable to ask what happen if we instead consider the constraint $\langle \mu, \phi \rangle \geq r$. Therefore, an investor wishes a portfolio return which is at least equal to r. Suppose, that $\langle \mu, \phi \rangle > r$ holds. By the slackness condition in the KKT theorem, the corresponding multiplier for this constraint is zero. This simplifies the optimization problem and we get for the optimal policy in this model

$$\hat{\phi}^* = \frac{V^{-1}\mu}{c} \ . \tag{24}$$

With this portfolio, an arbitrary optimal portfolio $\phi^*(r)$ can be written in the form

$$\phi^*(r) = \nu(r)\phi_m^* + (1 - \nu(r))\hat{\phi}^* \ . \tag{25}$$

The function $\nu(r)$ can be determined by the representation of $\phi^*(r)$ in Proposition 8.

We already found two decompositions of the optimal minimum variance portfolio in two portfolios. In fact, we show that any minimum variance portfolio $\phi^*(r)$ can *always* be written as a combination of two linearly independent minimum variance portfolios. Since for any portfolio, there exists another one such that a weighted combination is equal to the optimal portfolio, the decomposition is called *the mutual fund theorem* in the literature. The next proposition summarizes the discussion.

Proposition 15 (Mutual Fund Theorem). *Any minimum variance portfolio can be written as a combination of the global minimum variance portfolio and the portfolio $\hat{\phi}^*$. Furthermore, any minimum variance portfolio is a combination of any two distinct minimum variance portfolios.*

The second statement means that we can replace the global minimum variance portfolio and $\hat{\phi}^*$ by any other combination of minimum variance portfolios.

Proof. We give two proofs, first an explicit one and the second proofs exploits the convexity structure.
The KKT conditions in the proof of Proposition 8 imply that any solution of the optimization problem is of the form

$$\phi = \lambda_1 V^{-1}e + \lambda_2 V^{-1}\mu \ .$$

The first term is proportional to the global minimum variance portfolio while the second one is proportional to $\hat{\phi}^*$. This proves the first claim. To prove the second claim, assume that ϕ^1 and ϕ^2 are two minimum variance portfolios. Then, they can be spanned by the global minimum variance portfolio and the portfolio $\hat{\phi}^*$ due to the first part of the proposition as follows

$$\phi^i = (1 - a^i)\phi^*_m + a^i \hat{\phi}^* \ , \ i = 1, 2 \ .$$

A solution ϕ^* of the minimum-variance problem can then be written as

$$\phi^* = \frac{\lambda_1 b + a^2 - 1}{a^2 - a^1} \phi^1 + \frac{1 - a^1 - \lambda_1 b}{a^2 - a^1} \phi^2 \tag{26}$$

where the multiplier λ_1 is given in (10). The coefficients of the above representation add up to 1. This proves the claim.

Using the result of Chapter **??**, Proposition **??**, the *value function* $\sigma^2(r)$ of the model $\mathcal{M}$ can be written in the form

$$\sigma^2(z) = \inf_{\phi} \{ \langle \phi, V\phi \rangle | \ \phi \in S \ , \ G(\phi) = z \} \tag{27}$$

with $z = (r, 1)'$ and $G(\phi) = (\langle \mu, \phi \rangle, \langle e, \phi \rangle)'$. Since all assumptions of Proposition **??** are fulfilled, $\sigma^2(z)$ is a convex, increasing function for $r \geq r_*$. Therefore, for any portfolio $\phi^*_1, \phi^*_2 \in \sigma^2(z)$ and for any $0 \leq \lambda \leq 1$ there exist a portfolio ϕ^*_3 in $\sigma^2(z)$ which is a convex combination of the two portfolios, i.e.

$$\lambda \phi^*_1 + (1 - \lambda)\phi^*_2 = \phi^*_3 \ .$$

□

Corollary 16. *The efficient frontier is a convex set.*

Proof. Exercise. □

The practical implication are the following ones. Suppose that an investor wishes to invest optimally according to the mean-variance criterion. A possibility is to buy the assets such as prescribed by the portfolio $\phi^*(r)$. But if the number of assets is large this might be impossible to achieve since he is not wealthy enough (consider for example the price of Swiss stocks!). The decomposition property helps investor facing the described problem. Suppose, that there exist two funds with the characteristic of the two portfolios in the decomposition (25). Then the investor only has to invest in these two funds to buy the optimal portfolio.
The fact that $\sigma^2(z)$, $z = (r, 1)'$, is a convex, increasing function for $r \geq r_*$ can be used to highlight the trade-off between risk and return. By defintion $\sigma^2(z)$ is increasing if $z_1 \geq z_2$ implies $\sigma^2(z_1) \geq \sigma^2(z_2)$. But $z_1 \geq z_2$ is equivalent to $r_1 \geq r_2$, i.e. the expected return of a portfolio $\phi^{(1)}$ is larger than that one of $\phi^{(2)}$, implies $\sigma^2(z_1) \geq \sigma^2(z_2)$, i.e. the optimal risk of portfolio $\phi^{(1)}$ is also larger than that one of the portfolio $\phi^{(2)}$.
We finally mention the asymptotic formula

$$\lim_{r \to \infty} \left(\sigma(r) - r\sqrt{\langle \phi^*_0, V\phi^*_0 \rangle} \right) = -r_* \sqrt{\langle \phi^*_0, V\phi^*_0 \rangle} \ . \tag{28}$$

Prove this formula by direct computation.
The covariance properties of minimum variance portfolios are summarized in the following proposition.

Proposition 17. *The covariance of two minimum variance portfolios*

$$\phi^*_i = (1 - a_i)\phi^*_m + a_i \hat{\phi}^* \ , \ i = 1, 2$$

is given by

$$cov(R^{\phi^*_1}, R^{\phi^*_2}) = \frac{1}{b} + \frac{a_1 a_2 \Delta}{bc^2} \ . \tag{29}$$

The covariance of the global minimum variance portfolio ϕ_m^* *with any minimum variance portfolio* ϕ^* *is*

$$cov(R^{\phi_m^*}, R^{\phi^*}) = \frac{1}{b} \ . \tag{30}$$

PROOF.

$$\begin{aligned}
cov(R^{\phi_1^*}, R^{\phi_2^*}) &= \langle \phi_1^*, V\phi_1^* \rangle \\
&= (1-a_1)(1-a_2)\langle \phi_m^*, V\phi_m^* \rangle + a_1 a_2 \langle \hat{\phi}^*, V\hat{\phi}^* \rangle + (1-a_1)a_2 \langle \phi_m^*, V\hat{\phi}^* \rangle \\
&+ a_1(1-a_2)\langle \hat{\phi}^*, V\phi_m^* \rangle \\
&= (1-a_1)(1-a_2)\frac{b}{b^2} + a_1 a_2 \frac{a}{c^2} + \frac{1-a_1)a_2 c + a_1(1-a_2)c}{ba} \\
&= (1-a_1)(1-a_2)\frac{1}{b} + a_1 a_2 \frac{a}{c^2} + \frac{a_1 + a_2 - 2a_1 a_2}{b} \\
&= \frac{1}{b} + \frac{a_1 a_2 \Delta}{bc^2} \ .
\end{aligned}$$

If we set $a_1 = 0$ the second claim follows. □

To close this section we consider an example. Consider the case of two securities with the data $\mu_1 = 1, \mu_2 = 0.9$ and

$$V_{11} = 0.1, V_{22} = 0.15, V_{21} = V_{12} = -0.1 \ .$$

The expected return r is set equal to $r = 0.96$. Hence, the two assets are negatively correlated and if we neglect the correlation structure, asset 2 does not seems to be attractive since it possesses a lower expected return and a larger risk (measured by the variance). However, the negative correlation will induce that it can be advantageous to invest in the second asset too. First we consider the strategies $\phi_1 = (1, 0), \phi_2 = (\frac{1}{2}, \frac{1}{2})$. We get

$$var(R^{\phi_1}) = 0.1 \ , \ E(R^{\phi_1}) = 1 \ , \ var(R^{\phi_2}) = 0.0.0125 \ , \ E(R^{\phi_1}) = 0.95 \ .$$

Although ϕ_1 satisfies the expected return condition $r = 0.96$, the risk is much larger than for the strategy ϕ_2 which in turn does not satisfy the expected return condition. As a final strategy we consider ϕ_3 which is obtained by solving the optimization problem without imposing any restrictions on the expected return. It follows that $\phi_3 = (\frac{5}{9}, \frac{4}{9})$ is the searched portfolio and

$$var(R^{\phi_3} = 0.011 \ , \ E(R^{\phi_3}) = 0.955 \ .$$

Hence the risk is minimum but the expected return is smaller than $r = 0.96$. Finally, if we solve the full problem the optimal portfolio reads $\phi^* = (0.6, 0.4)$ and we get

$$var(R^{\phi^*} = 0.012 \ , \ E(R^{\phi^*}) = 0.96 \ .$$

Hence, 40 percent has to be invested in the not very attractive asset which again reflects the Markowitz phenomenon. For this portfolio, compared to the naive one ϕ_1, the variance is reduced drastically and the expected return is still acceptable.

3. Diversification

We consider diversification with respect to the number of assets N chosen in the portfolio problem. Diversification could also be considered with respect to criteria such as "sectorial diversification of the assets" for example. To start with, we consider two returns R_1, R_2 and we denote $\sigma_i = \sqrt{var(R_i)}$. It then follows

$$var(R^{\phi}) = var(\phi_1 R_1 + \phi_2 R_2) = (\phi_1\sigma_1 - \phi_2\sigma_2)^2 + \phi_1\phi_2\sigma_1\sigma_2(1 + \sigma_{12})$$

with

$$\sigma_{12} = \frac{cov(R_1, R_2)}{\sigma_1 \sigma_2} = \frac{E[R_1 R_2] - E[R_1]E[R_2]}{\sigma_1 \sigma_2} \ .$$

Hence, if $\phi_1 \sigma_1 = \phi_2 \sigma_2$ and $\sigma_{12} = -1$, risk vanishes

$$var(R^\phi) = 0 \ .$$

Therefore, if R_1 and R_2 are perfectly negatively correlated, we can choose ϕ_1, ϕ_2 such that $\phi_1 \sigma_1 = \phi_2 \sigma_2$ holds and the portfolio is then riskless. Clearly, the expected return will be small to. This examples suggests, that in building an investment portfolio ϕ one must invest in possibly many negatively correlated securities for reducting the variance of the portfolio. This rule is known as the *Markowitz phenomenon.*

To proceed, we first assume that the N returns $R_1, \ldots, R_N$ are *uncorrelated* and that there exist a constant c which is an upper bound for the risks, i.e. $var(R_i) \leq c$ for all i. Using elementary properties of the variance

$$var(\sum_{i=1}^N \phi_i R_i) = \sum_{i=1}^N \phi_i^2 var(R_i) \leq c \sum_{i=1}^N \phi_i^2$$

follows. If we suppose that the investment is equidistributed for all securities, i.e. $\phi = \frac{1}{N}$, then

$$var(\sum_{i=1}^N \phi_i R_i) \leq c \sum_{i=1}^N \phi_i^2 = \frac{c}{N} \to 0 \ , \ (N \to \infty).$$

This shows that to reduce the risk for uncorrelated securities under an equidistribution investment strategy the number of assets invested in has to be increased.

What can be said if we relax the condition that assets are uncorrelated? In fact most assets traded in stock exchanges are positively correlated. Again, we consider the case of an equidistribution strategy $\phi_i = \frac{1}{N}$. We then have

$$\begin{aligned} var\langle \phi, V\phi \rangle &= \sum_{i=1}^N \phi_i^2 var(R_i) + \sum_{i,j=1, i\neq j}^N \phi_i \phi_j cov(R_i, R_j) \\ &= (\frac{1}{N})^2 N \frac{1}{N} \sum_{i=1}^N var(R_i) + (\frac{1}{N})^2 (N^2 - N) \frac{1}{N^2 - N} \sum_{i,j=1, i\neq j}^N cov(R_i, R_j) \\ &= \frac{1}{N} \bar{\sigma}_N^2 + (\frac{1}{N})^2 (N^2 - N) c\bar{o}v_N \\ &= \frac{1}{N} \bar{\sigma}_N^2 + (1 - \frac{1}{N}) c\bar{o}v_N, \end{aligned}$$

where we defined the mean variance $\bar{\sigma}_N^2$ and the mean covariance $c\bar{o}v_N$. If the the mean variance is bounded, $\bar{\sigma}_N^2 \leq N$, and the mean covariance converges to a limit value $c\bar{o}v_\infty$ for $N \to \infty$, then

$$var(R^\phi) \to c\bar{o}v_\infty \ , \ (N \to \infty).$$

Hence, if $c\bar{o}v_\infty = 0$, using diversification in N implies that the risk can be arbitrarily reduced. Typically, $c\bar{o}v_\infty$ does *not* converges to zero for an increasing number of assets. The risk remaining risk

$$c\bar{o}v_\infty = \lim_{N \to \infty} c\bar{o}v_N$$

can not be diversified and is called the systematic (market) risk. The diversifiable risk $\frac{1}{N} \bar{\sigma}_N^2$ is the *unsystematic risk.*

4. Mean-variance analysis with a riskless asset

We assume that there exist a riskless asset B_t with return $\mu_0 = B(1) - 1$. Furthermore, there are N risky assets in the economy and we assume that the mean-variance criterion is used to select portfolios. All other properties of the economy and of the financial market are left unchanged. Also, the technical assumption are assumed to hold true, i.e. the covariance matrix is strictly positive, the vectors $e = (1, \dots, 1) \in \mathbf{R}^N$ and $\mu = (\mu_0, \mu_1, \dots \mu_N)$ are linearly independent, where μ_0 is the (expected) return of the riskless asset. The optimization problem then reads

$$\begin{aligned} \min_{\phi} \quad & \frac{1}{2}\langle \phi, V\phi\rangle \qquad \mathcal{M}_R \\ \text{s.t.} \quad & S_R = \{\phi \in \mathbf{R}^N | \ \langle e, \phi\rangle = 1 - \phi_0 \\ & \langle \mu, \phi\rangle = r - \mu_0\phi_0 \ \} \end{aligned} \tag{31}$$

The solution of this problem follows that one of Proposition 8, it is even simpler. We assume that the covariance matrix is strictly positive and that the vectors μ and $\mu_0 e$ are linearly independent. The results are summarized in the next proposition.

Proposition 18. *If the same assumptions as in Proposition 8 hold, then the model $\mathcal{M}_R$ possesses the solution*

$$\begin{aligned} \phi^* &= \lambda^* V^{-1}(\mu - \mu_0 e) \\ \lambda^* &= \frac{r - \mu_0}{a - 2\mu_0 c + \mu_0^2 b} =: \frac{r-\mu_0}{\Delta_R}. \end{aligned} \tag{32}$$

The locus of minimum variance portfolios is given by

$$\sigma_R(r) = \pm\frac{r - \mu_0}{\sqrt{\Delta_R}} \ . \tag{33}$$

The proof of the proposition is left to the reader since it is similar than the proof with risky assets only. It follows, that the in $\sigma(r), r)$-space the locus of the minimum-variance set are again (degenerated) hyperbolas, i.e. straight lines. The next proposition characterizes minimum variance portfolios and specifies a "canonical" representation of minimum variance portfolios.

Proposition 19. (1) *All minimum variance portfolio are combinations of any two linearly independent minimum variance portfolios.*
(2) *The minimum variance portfolio in the model $\mathcal{M}_R$ with zero investment in the riskless asset ($\phi_0 = 0$) is given by*

$$\phi^T = \frac{1}{c - \mu_0 b} V^{-1}\mu - \frac{\mu_0}{c - \mu_0 b} V^{-1} e \ . \tag{34}$$

This portfolio is called the tangency or market portfolio. The tangency portfolio is also an efficient portfolio in the model $\mathcal{M}$.
(3) *The efficient portfolios of the model $\mathcal{M}_R$ on the branch $r = \mu_0 + \sigma\sqrt{\Delta_R}$ is tangent to the efficient frontier of the model $\mathcal{M}$.*

Proof. The proof of 1. directly follows from the convexity of the set

$$\sigma_R^2(z) = \inf_{\phi}\{\langle \phi, V\phi\rangle | \ \phi \in S_R \ , \ G_R(z) = z\} \ ,$$

with G_R and z the appropriate matrix and vector of the problem $\mathcal{M}_R$. To prove 2., we note that the risky assets investments add up to 1 and multiplication of (32) by e from the left then implies

$$1 \ = \ \lambda^* \langle e, V^{-1}(\mu - \mu_0 e)\rangle \Longrightarrow \lambda^* = \frac{1}{c - \mu_0 b} \ .$$

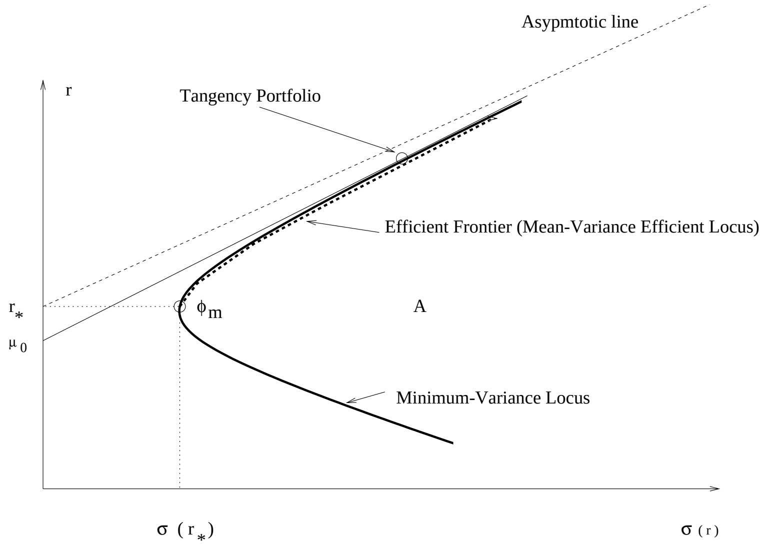

FIGURE 4. Mean-variance illustrations in the $(\sigma(r), r)$-space for the case with and without a riskless asset, respectively.

Substituting this value in (32) proves 2. To prove that the tangency portfolio is efficient, we first calculate the expected return of the tangency portfolio. It follows

$$r = \langle \mu, \phi^T \rangle = \frac{a - \phi_0 c}{c - \phi_0 b} \ .$$

Inserting this in the optimal minimum variance portfolio expression of Proposition 8 implies

$$\begin{aligned} \phi^* &= \frac{a - c\frac{a-\phi_0 c}{c-\phi_0 b}}{\Delta} V^{-1} e + \frac{-c + b\frac{a-\phi_0 c}{c-\phi_0 b}}{\Delta} V^{-1} \mu \\ &= \frac{1}{c - \mu_0 b} V^{-1} \mu - \frac{\phi_r}{c - \mu_0 b} V^{-1} e = \phi^T \ , \end{aligned}$$

after some algebra. To prove 3., we first derive the mean-variance relationship of the classical model $\mathcal{M}$ (equation (13)),

$$\frac{d\sigma(r)}{dr} = \frac{1}{\sqrt{\Delta(br^2 - 2rc + a\rangle)}} (br - c) \ .$$

If we evaluate the derivative at the point of the tangency portfolio, we get

$$\frac{d\sigma^T(r)}{dr} = \left(b\mu_0^2 - 2\mu_0 c + a\right)^{\frac{1}{2}}$$

which is the inverse of the slope of the former derivative at $\mu_0 = r$. □

Figure 4 compares the different loci in the $(\sigma(r), r)$-space in the case with and without riskless asset, respectively.

Figure 4 also indicates the following holds true, which in fact can easily be proven:

PROPOSITION 20. *Assume the models $\mathcal{M}$ and $\mathcal{M}_R$ be given and that $\mu_0 \neq r^*$. Then the tangency portfolio is element of the efficient frontier iff $\mu_0 < r^*$. The tangency portfolio is element of the inefficient part of the minimum variance locus iff $\mu_0 > r^*$. If $\mu_0 = r^*$, the tangency portfolios does not exists for any finite risk and return. The portfolios are then given by the intersection of the asymptotic portfolios and the efficient set at infinity.*

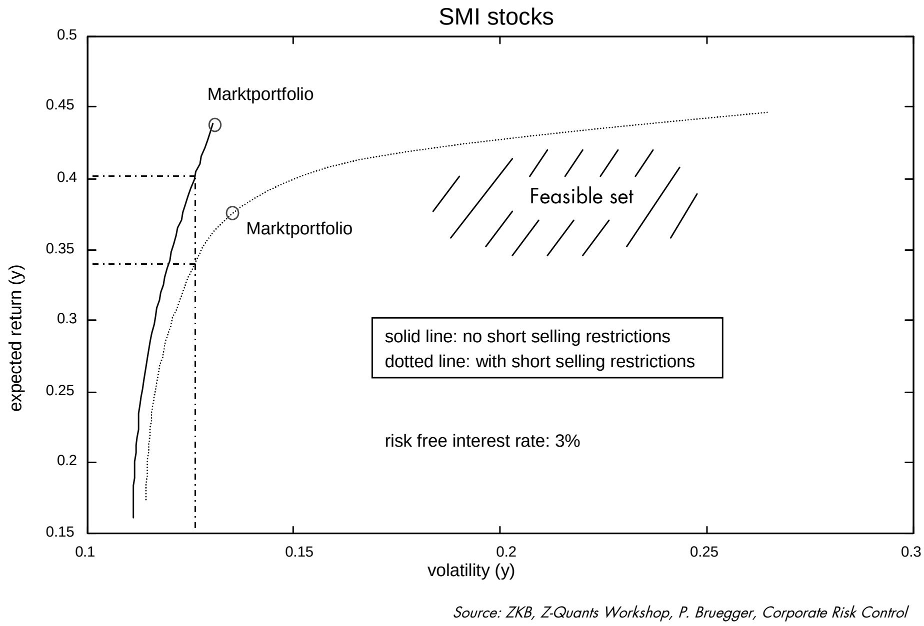

FIGURE 5. The efficient frontiers for two Markowitz-type models are shown: The Markowitz model without and with short selling restrictions. Data: SMI, daily, year 2000

We have proven that as in the case of risky-assets only any minimum variance portfolio is a linear combination of two distinct minimum variance portfolios. Contrary to the risky-asset only case, there is a natural choice of the mutual funds: The riskless asset and the fund with no riskless asset, i.e. the tangency portfolio ϕ^T. A necessary and sufficient condition - analogous to the risky asset only case in Proposition 14 - can be given.

PROPOSITION 21. *Assume that the technical assumptions hold in the model $\mathcal{M}_R$. For a portfolio ϕ being an efficient portfolio necessary and sufficient is*

$$E[R_i^\phi] = \mu_0 + cov(R_i, R^\phi)\frac{E[R^\phi] - \mu_0]}{\sigma^2(R^\phi)} \ , \ i = 1, \ldots, N \tag{35}$$

with $E[R^\phi] - \mu_0 > 0$.

Since the proof is analogous to the proof of Proposition 14, it is omitted.

5. Numerical examples

Figures 5, 5 and **??** illustrate the Markowitz model for the Swiss Market Index (SMI). For the analysis, daily data of the year 2000 where used. Figures 5 illustrates the fact that imposing linear restrictions, such as short selling restrictions in this example, implies that the feasible set of portfolios with restrictions is a subset of the feasible set without restrictions. Hence, for the same risk, the expected return of the efficient portfolio without restrictions dominates that one with short selling restrictions. Figures 5 and **??** compare the optimal portfolio weights for

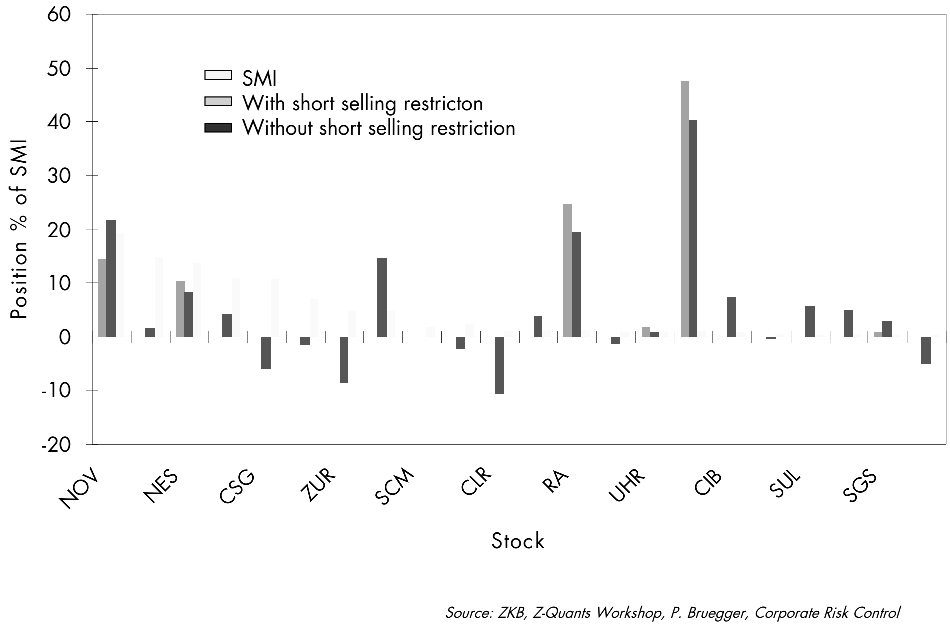

FIGURE 6. Portfolio weights for the SMI compared with the efficient portfolio weights in the Markowitz model, where once short selling was allowed and in the second calculation short selling was forbidden. Data: SMI, daily, year 2000

the SMI with and without short selling restrictions and the weights how they are used at the end of March 2001 in the index SMI. A first glimpse shows that the Markowitz model hardly can called an appropriate model for the SMI. We see that two securities, which contribute to the SMI with about 1 percent, indeed have in the case with no short selling conditions a weight of 19.45 and 40.21 percent respectively. These two assets, Rentenanstalt and Baloise Insurance Company, both faced "abnormal" events in the year 2000 where the data for the mean-variance are considered. First, there was strong rumor in the market that Rentenanstalt was taken over by Generali. This led to the usual overreactions of market participants. Baloise contrary faced a strong demand for its assets due to the heavy investment of the BZ bank. This two examples indicate that for the mean-variance analysis stocks which are under "regular" trading should be considered only. Figure 5 compares the SMI weights with the mean-variance with where once the two assets are incorporated and deleted (with weight equal to zero), respectively. If we compare the optimal values in the two setups, we see that the distribution of about 60 percent of wealth from the two deleted assets, is not **not** uniformally on the remaining assets. This non-linearity is due to the new correlations between the SMI assets. Furthermore, if we calculate the difference between the optimal weight for each asset to its SMI weight, sum the differences and take the square root, it follows that the error decreased by a 30 percent if the two extreme assets are deleted. Another improvement is achieved if we consider more than one year to determine the value

Stock	SMI-Weighting	Mean Variance [	Mean Variance [%	delta normal	delta normal VaR [%]	Return year 2000
	[%]	Short Selling Restrictions	No Short Sell Restrictions	Short Selling	No Short Sell	[%]
NOV	19.24	14.47	21.57	14.47	21.57	30
ROG	14.88	-	1.80	-	1.80	-5
NES	13.78	10.52	8.17	10.52	8.17	35
UBS	11.01	-	4.40	-	4.40	30
CSG	10.63	-	-6.05	-	-6.05	0
ABB	6.86	-	-1.52	-	-1.52	-5
ZUR	4.78	-	-8.63	-	-8.63	10
RUK	4.83	-	14.59	-	14.59	25
SCM	1.90	-	-0.09	-	-0.09	-35
ADE	2.38	-	-2.33	-	-2.33	-15
CLR	1.07	-	-10.68	-	-10.68	-20
HOL	1.26	-	3.98	-	3.98	-5
RA	1.27	24.68	19.45	24.68	19.45	50
ALUS	0.78	-	-1.38	-	-1.38	-20
UHR	0.82	2.02	0.90	2.02	0.90	15
BAL	0.99	47.46	40.21	47.46	40.21	40
CIB	0.81	-	7.49	-	7.49	-5
LON	0.68	-	-0.43	-	-0.43	0
SUL	0.48	-	5.73	-	5.73	10
UHR	0.69	-	4.93	-	4.93	15
SGS	0.40	0.85	2.96	0.85	2.96	10
SWS	0.47	-	-5.08	-	-5.08	-15

Source: ZKB, Z-Quants Workshop, P. Bruegger, Corporate Risk Control

FIGURE 7. Portfolio weights for the SMI compared with the efficient portfolio weights in the Markowitz model, where once short selling was allowed and in the second calculation short selling was forbidden. In this table also data for the delta-normal approach to mean-Value-at-Risk portfolio optimization are shown. The results will be used in the next chapter. Data: SMI, daily, year 2000

of the model parameters. In Figure 5 efficient frontiers for the SMI with two and one year daily data are shown. It follows it we compare the optimal investments for these two cases without the two extreme assets, that the error is reduced by another 15 percent.

In summary, this examples indicate that

(1) If assets facing extreme events (mergers, take overs) are part of the data used to estimate the model parameter, they should not be considered.
(2) The longer a time series the "more stable" the Markowitz model seems to be. But if the series is to long such that in the mean while economy underwent structural changes, parameters which does not reflect the present economy are estimated.
(3) If assets are deleted after a first optimization, their percentages of wealth can not be distributed symmetrically on the remaining assets. The redistribution is non-linear due to the changes correlations and has to be calculated again.

As a final example, we reconsider the example with three risky instruments: The Standard& Poor's index 500, US Government Bonds and a US Small Cap Index.

	SMI Weights	Optimal SMI	Without Rentenanstalt and Baloise	Deviation SMI - SMI optimal	Deviation SMI - SMI opt. without RA and BA
NOV	19.24	21.57	35.09	5.42	251.19
ROG	14.88	1.80	6.55	171.19	69.46
NES	13.78	8.17	19.55	31.49	33.28
UBS	11.01	4.40	5.41	43.69	31.36
CSG	10.63	-6.05	-5.76	278.31	268.72
ABB	6.86	-1.52	0.14	70.27	45.20
ZUR	4.78	-8.63	-3.48	179.77	68.19
RUK	4.83	14.59	17.95	95.16	172.01
SCM	1.90	-0.09	-0.97	3.96	8.23
ADE	2.38	-2.33	-0.41	22.19	7.79
CLR	1.07	-10.68	-14.05	138.01	228.54
HOL	1.26	3.98	5.54	7.38	18.29
RA	1.27	19.45	0.00	330.69	1.60
ALUS	0.78	-1.38	-1.48	4.65	5.09
UHR	0.82	0.90	0.53	0.01	0.08
BAL	0.99	40.21	0.00	1538.50	0.97
CIB	0.81	7.49	3.24	44.65	5.91
LON	0.68	-0.43	12.10	1.24	130.38
SUL	0.48	5.73	8.69	27.59	67.44
UHR	0.69	4.93	7.01	18.02	40.00
SGS	0.40	2.96	6.71	6.55	39.81
SWS	0.47	-5.08	-2.37	30.76	8.04
			Square-root of quadratic error	**55.2221301**	**38.75022809**

FIGURE 8. Comparing the efficient frontier for the SMI with one-year versus two-years daily data.

On a monthly basis the expected returns are

$$\mu = (0.0101, 0.00435, 0.0137)' = (\text{ Standard\& Poor, US Gov. Bonds, Small Cap })'$$

and the covariance matrix V is

$$V = \begin{pmatrix} 0.003246 & 0.000229 & 0.004203 \\ 0.000229 & 0.000499 & 0.000192 \\ 0.004203 & 0.000192 & 0.007640 \end{pmatrix} .$$

The optimal strategy then is

$$\phi^* = (0.4520, 0.1155, 0.4324) .$$

We next assume that the parameters of the model, which are estimated using historical data, will in effect be different in the future by a small amount. We consider the impact of 5 percent misspecified parameters in various scenarios.

(1) Scenario A: All return are 5 percent higher than the estimated ones.
(2) Scenario B: The covariance matrix entries between the Standard& Poor and the US Gov. Bonds are are 5 percent higher than the estimated ones.
(3) Scenario C: The covariance matrix entries between the Standard& Poor and the US Gov. Bonds are are 5 percent smaller than the estimated ones.
(4) Scenario D: All covariance matrix entries are at random perturbed by ± 5 percent.
(5) Scenario E: The variance entry for the Standard& Poor is 5 percent higher than the estimated one.

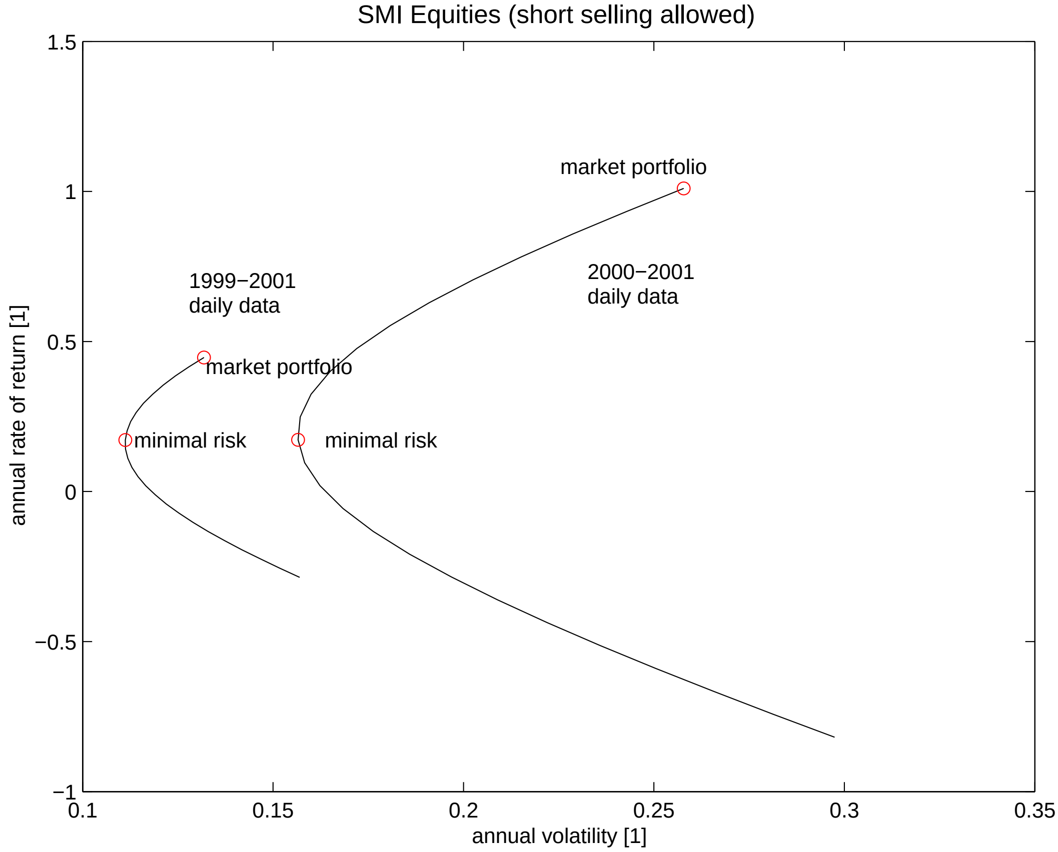

FIGURE 9. Comparing the efficient frontier for the SMI with one-year versus two-years daily data.

With this scenarios, the following differences to the original optimal strategy ϕ^* result:

Scenario	Difference in investment
A	$(0.0289, -0.0671, 0.0381)$
B	$(-0.0007, 0.0002, 0.0004)$
C	$(0.0006, -0.0002, -0.0004)$
D	$(-0.1819, 0.06994, 0.1120)$
E	$(-0.0926, 0.03560, 0.0570$

The differences in investment are not relative changes with respect to the original strategy but absolute ones! Hence, the number -0.1819 for the Standard& Poor in scenario D means that instead of investing 45.20 percent of wealth in this index, 63.39 percent have to be optimally invested if the whole covariance matrix is perturbed at random by ± 5 percent changes. It follows, that the effect various from negligible to a more than a 50 percent changement of the original optimal investment.
In summary, small variations of the input parameters in the Markowitz model - returns and covariance matrix - can have a very large range of implications on optimal investment: From negligible changes to a radically different new strategy.
Hence, a further basic issue is to determine the model parameters correctly. But, what does this mean? It is intuitively clear, that the usual approach of historical estimation of these parameters is only suitable if market circumstances are not

changing - and as we have seen, for some parameter values they are not allowed to change even a little. The Markowitz model makes no suggestions how these parameter are to be found.

6. Mean-variance analysis in complete markets

We consider the case of two assets. It then follows from finance theory, that financial market is complete if and only if the number of states equals the number of assets with linearly independent payoff. Therefore, we assume that there are two states and the (objective) probability law is the binomial law with probabilities p and $1-p$ for the state realizations, respectively. The expected return relation is

$$\langle \mu, \phi \rangle = \mu_1 \phi_1 + \mu_2 \phi_2 = r \ .$$

The standard deviation can be written in the form

$$\sigma(\phi) = \pm\sqrt{p(1-p)}\left((V_{11}-V_{21})\phi_1 + (V_{12}-V_{22})\phi_2\right) \ .$$

The iso-risk contours $\sigma(\phi) = \bar{\sigma}$ are linear in the trading strategies as the iso-expected return are. Solving these two linear equations for the trading strategy again implies a *linear* relationship between return and risk. This linearity is also maintained in the full optimization program, i.e. the relationship in the (σ, μ)- space is linear as for the case with a riskless asset. Since the two linear rays intersect at the point $(0, r)$, the variance of the global minimum variance portfolio is zero (as in the case with a riskless asset).

We prove the claims in the next step. The formal model reads

$$\begin{aligned} \min_{\phi} \quad & \frac{1}{2}\langle \phi, V\phi \rangle = \frac{1}{2}p(1-p)((V_{11}-V_{21})\phi_1 + (V_{12}-V_{22})\phi_2)^2 \qquad (\mathcal{M}_{comp}) \quad (36) \\ s.t. \quad & \phi_1 + \phi_2 = 1 \\ & \mu_1\phi_1 + \mu_2\phi_2 = r \ . \end{aligned}$$

Replacing ϕ_2 using the normalization condition, the first order condition solved with respect to ϕ_1^* implies

$$\begin{aligned} \phi_1^* &= \frac{\lambda(\mu_1-\mu_2)}{p(1-p)\Delta_{comP}^2} - \frac{V_{12}-V_{22}}{\Delta_{comp}} \\ &=: \lambda x - y \ . \end{aligned} \qquad (37)$$

with

$$\Delta_{comp} = V_{11} - V_{21} + V_{22} - V_{12} \ .$$

Using the expected return condition, the multiplier λ is

$$\lambda = \frac{p(1-p)\Delta_{comp}^2}{\mu_1-\mu_2}\left(\frac{r-\mu_2}{\mu_1-\mu_2} + \frac{V_{12}-V_{22}}{\Delta_{comp}}\right) \ . \qquad (38)$$

Inserting (37) and (38) into the value function implies

$$\begin{aligned} \sigma_{comp}^2(r) &= \min_{\phi}\frac{1}{2}\langle\phi, V\phi\rangle = \frac{1}{2}p(1-p)((V_{11}-V_{21})\phi_1 + (V_{12}-V_{22})\phi_2)^2 \qquad (39) \\ &= \frac{1}{2}p(1-p)\Delta_{comp}^2(\lambda x \underbrace{-y + \frac{V_{12}-V_{22}}{\Delta_{comp}}}_{=0})^2 = \frac{1}{2}p(1-p)\Delta_{comp}^2(\lambda x)^2. \end{aligned}$$

Therefore,

$$\sigma_{comp}(r) = \pm\sqrt{\frac{1}{2}p(1-p)}\Delta_{comp}\lambda x \qquad (40)$$

which proves that the minimum variance efficient set in the (σ, r)-space are straight lines. Inserting the explicit expressions we get

$$\sigma_{comp}(r) \quad = \quad \pm\sqrt{\frac{1}{2}p(1-p)}\Delta_{comp}\left(\frac{r-\mu_2}{\mu_1-\mu_2}+\frac{V_{12}-V_{22}}{\Delta_{comp}}\right) \ . \tag{41}$$

It follows that $\sigma_{comp}(r) = 0$ if either $p = 0$ or $p = 1$, or for $0 < p < 1$ the expected return condition

$$r^* = \frac{\mu_1(V_{22}-V_{12})+\mu_2(V_{11}-V_{21})}{\Delta_{comp}}$$

implies that the variance of the optimal portfolio is zero. If the two assets are negatively correlated, $r^* > 0$ follows. Hence, in this case we can achieve a zero-variance optimal portfolio and expect a positive return.

The result in this section shows a basic difference between complete and incomplete financial markets. In complete financial markets with risky assets only it is possible to eliminate risk completely and to expect a positive return. This is not possible in the case of incomplete financial markets. The locus in this example are qualitatively not different to the incomplete case since straight lines are just the stretched out hyperbolas of the incomplete market.

7. Mean-variance analysis and no arbitrage

We shortly recall some notions from mathematical finance. The Markowitz model is based on a probability space $(\Omega, \mathcal{F}, P)$ with P the objective probability of the security prices.

Definition 22. *Let Q be an equivalent probability measure to P. If all discounted prices at time $t = 1$ ($\frac{S_j(1)}{B(1)}$), $j = 1, \ldots, N$, are Q-martingales, we call Q a risk neutral probability (RNP)*[7]

The easy to prove part of the Fundamental Theorem of Finance implies that the existence of a RNP implies the absence of arbitrage. This means,

Definition 23. *An arbitrage is a portfolio ψ such that starting from zero initial wealth $V_0^\psi = 0$ with certainty no losses occur, i.e.*

$$P(V_1^\psi \geq 0) = 1 \ , \ P(V_1^\psi > 0) > 0 \ .$$

We assume in the sequel that the primitive financial market is free of arbitrage. If ψ is normalized portfolio and Q a RNP we calculate the discounted value process (wealth process) using the Bookkeeping Proposition 1:

$$\begin{aligned}
& E_Q^\psi[\frac{V_1}{B_1}|\mathcal{F}] = E_Q^\psi[\frac{V_0+G}{B_1}|\mathcal{F}] \\
= \quad & \frac{V_0}{B_1} + E_Q^\psi[\frac{\psi_0 r + \sum_j \psi_j \Delta S_j(1)}{B_1}|\mathcal{F}] = \frac{\psi_0 B(0) + \psi_0 r}{B_1} + \frac{\sum_j \psi_j S_j(0)}{B_1} \\
= \quad & \psi_0 \cdot 1 + \sum_j \psi_j S_j(0) = \frac{V_0}{B_0} = V_0 \ .
\end{aligned}$$

Hence we proved

Proposition 24. *If Q is a RNP, for all strategies ψ the discounted portfolio process is a Q-martingale.*

[7] Sometimes Q is also called a risk adjusted measure in the literature.

The remarkable fact is that the strategy chosen does not matter in the proposition. Since

$$\tilde{S}_j(1) - \tilde{S}_j(0) = S_j(0)\frac{R_j - R}{1+R}$$

it follows at once

PROPOSITION 25. *If $Q(\omega) > 0$ for all $\omega \in \Omega$ is a probability measure, then Q is a RNP iff*

$$E_Q[\frac{R_j - R}{1+R}] = 0 \ , \ j = 1, \ldots, N \ . \tag{42}$$

If we assume that the interest rate R_0 is deterministic, (42) simplifies to

$$E_Q[R_j] = R_0 \ , \ j = 1, \ldots, N \ . \tag{43}$$

This condition has a beautiful interpretation. Suppose that markets are free of arbitrage and that there exists a deterministic interest rate. Then the expected rate of return of all risky assets are equal and are given by the risk free rate under the risk neutral probability. We finally recall that a contingent claim X is attainable if there exists a portfolio ϕ such that

$$X(\omega) = V^{\phi}(\omega) \ , \ \forall \omega \in \Omega$$

holds. The next proposition summarizes the impact of no-arbitrage on the relationship between different portfolios.

PROPOSITION 26. *Suppose that the primitive financial market is free of arbitrage and that there exists a risk free security with interest rate $R_0 = \mu_0$. For Q a RNP, let*

$$L(\omega) = \frac{Q(\omega)}{P(\omega)}$$

be the state price vector (or state price density) which possesses a first moment. Suppose further that the random variable (contingent claim)

$$X = a + bL \ , \ a, b \in \mathbf{R}, b \neq 0$$

is attainable for some normalized portfolio $\hat{\phi}$ with return $R^{\hat{\phi}}$. Let R^{ϕ} be the return of an arbitrary portfolio ϕ. Then

$$E[R^{\phi}] - \mu_0 = \frac{cov(R^{\hat{\phi}}, R^{\phi})}{var(R^{\hat{\phi}})}(E[R^{\hat{\phi}}] - \mu_0) \ . \tag{44}$$

PROOF. To start with, we calculate

$$\begin{aligned} cov(R_j^{\phi}, L) &= E[R_j^{\phi} L] - E[R_j^{\phi}]E[L] = E[R_j^{\phi} L] - E[R_j^{\phi}] \\ &= E_Q[R_j^{\phi}] - E[R_j^{\phi}] = R_0 - E[R_j^{\phi}] \ , \forall j. \end{aligned}$$

Hence,

$$-cov(R^{\phi}, L) = E[R^{\phi}] - R_0 \tag{45}$$

follows. Let $\hat{\phi}$ be the strategy such that the contingent claim $X = a + bL$ is attainable. Then

$$V_1^{\hat{\phi}} = a + bL \ \Rightarrow \ V_0^{\hat{\phi}}(1 + R^{\hat{\phi}}) = a + bL \ .$$

Hence, solving for L implies

$$L = \frac{V_0^{\hat{\phi}}(1 + R^{\hat{\phi}})}{b} - \frac{a}{b} \ .$$

Inserting this expression into (45) leads to

$$cov(R^\phi, L) = \frac{V_0^{\hat{\phi}}}{b} cov(R^\phi, R^{\hat{\phi}}) \ . \tag{46}$$

Therefore,

$$E[R^\phi] - R_0 = -\frac{V_0^{\hat{\phi}}}{b} cov(R^\phi, R^{\hat{\phi}}) \tag{47}$$

follows. To eliminate initial wealth V_0, we set $\phi = \hat{\phi}$ in the last equality which implies

$$E[R^{\hat{\phi}}] - R_0 = -\frac{V_0^{\hat{\phi}}}{b} var(R^{\hat{\phi}}) \ . \tag{48}$$

Solving (48) with respect to initial wealth and inserting the result in (47) proves the claim. □

If we compare Proposition 26 with Proposition 21, where a characterization of efficient portfolios in the Markowitz model with a riskless asset is given, an astonishing similarity shows up. Under technical conditions, in the later proposition a portfolio is efficient iff

$$E[R_i^\phi] - R_0 = cov(R_i, R^\phi) \frac{E[R^\phi] - \mu_0]}{\sigma^2(R^\phi)} \ , \ i = 1, \ldots, N \tag{49}$$

with $E[R^\phi] - \mu_0 > 0$ holds. Although the functional form is the same in both approaches there significant differences in the underlying assumptions. First, an optimization problem was solved in one approach while the no-arbitrage notion was used in the other one. Second, in the optimization approach an affine relationship between the return of each portfolio component with the whole portfolio is equivalent to the efficiency of the portfolio in the second approach an affine relationship of the state price density and a portfolio to hedge this claim enters.

Why does such seemingly different approaches lead to such similar results? The answer will be given in the Chapter "Economic foundations".

8. Mean-variance analysis under investment restrictions

In the classical Markowitz model, the investment is unrestricted. That is, for example there are no short-sale restrictions.
This situation is highly stylized. In real investment problems, restrictions are often imposed either by regulatory authorities or there are internal restrictions in the firm which are to be respected. Restrictions can be imposed in various respects:

(1) Limitations in the risk exposure. For example a number $\epsilon > 0$ is exogenous given such that $\langle \phi, V\phi \rangle \leq \epsilon$. Another famous example is a limitation of the Value at Risk (VaR) of a portfolio of a trading unit. This latter restriction is more fundamental than the prior one since the risk measure of the Markowitz model (variance) is exchanged with a new measure for the risk exposure (VaR). Hence, imposing VaR-restrictions means to chose a new model.
(2) Diversification restrictions. An example of such a restriction is for example that the percentage amount of a portfolio invested in Swiss Market Blue Chips has not to be lower than 10 percent and is to be smaller than 30 percent.

(3) Liquidity restrictions are very important for the banking firm. Typically, this kind of restrictions has to be addressed in dynamic models of portfolio selection.

In this section we only consider diversification restrictions of the form

$$a \le \phi \le b \ , \ a, b, \phi \in \mathbf{R}^N \ . \tag{50}$$

In other words, the fractions of wealth invested in the securities has to lie within pre specified bounds. We assume that the two conditions

$$\sum_{j=1}^{N} a_j \le 1 \ , \ \sum_{j=1}^{N} b_j \ge 1 \tag{51}$$

hold. The first one is necessary for the portfolio problem to have a solution and the second one assures that total wealth will be invested. The optimization problem then reads

$$\begin{aligned} \min_{\phi} \quad & \frac{1}{2}\langle \phi, V\phi \rangle \qquad (\mathcal{M}_{rest}) \\ s.t. \quad & S_{rest} = \{\phi \in \mathbf{R}^N | \ \langle e, \phi \rangle = 1 \ , \ \langle \mu, \phi \rangle = r \ , \ a \le \phi \le b \ , \ a, b \in \mathbf{R}^N\} \ . \end{aligned} \tag{52}$$

Since the set S is an intersection of halfspaces and hyperplanes, it is a convex set and the program is itself convex. The constraints $a \le \phi \le b$ look very innocent. But we will see in the rest of this section that this not the case both from the mathematical and economic point of view. While the mathematical difficulties can be overcome by using computers we will show in examples that the economic consequences of such restrictions depend on an essential non-trivial way on *all* parameters of the problem. It is therefore doubtful whether regulatory authorities are aware of their consequence by imposing such innocent looking restrictions.

The KKT theorem still applies since the feasible set is still convex as an intersection of 2 hyperplanes and $2N$ half-spaces. We introduce the block matrices

$$D_1 = \text{diag}\ (\lambda_3, \dots, \lambda_{N+2}) \ , \ D_2 = \text{diag}\ (\lambda_{N+3}, \dots, \lambda_{2N+2}) \ .$$

The numeration starts with 3 since λ_1, λ_2 are used for the normalization - and the return condition, respectively. Since for each security there are two constraints, we have $2N$ multipliers. The matrix D_1 (D_2) summarizes the multipliers for the a-constraints (b-constraints).

$$\begin{aligned} 0 \ = \ & V\phi - \lambda_1 e - \lambda_2 \mu - D_1 a + D_2 b \\ & \langle e, \phi \rangle = 1 \ , \ \langle \mu, \phi \rangle = r \\ & \lambda_j \ge 0 \ , \ j = 3, 4, \dots 2N + 2 \\ & \lambda_{j,k}(a_k - \phi_k) = 0 \ , \ j = 3, \dots, N + 2 \\ & \lambda_{j,k}(\phi_k - b_k) = 0 \ , \ j = N + 3, \dots, 2N + 2 \ . \end{aligned} \tag{53}$$

The KKT theorem then implies that at most 3^N different cases have to be considered to find the optimal solution. More explicitly, a single case is analyzed as follows. Suppose that some of the assets bind and the others are not binding; this defines the particular case. Then, for the non binding assets, the respective multipliers for the upper and lower restriction have to be zero, else the slackness condition is violated. Then, since for the binding assets the portfolio contribution are known (the contributions of the restrictions), it seems that the number of variables which are to be found by solving the optimization program is reduced. This is wrong, since for each binding portfolio component we have to determine according to the KKT conditions whether the sign of the associated multiplier is positive. Therefore, we have to solve a system with the same dimensionality as in the unrestricted case. Suppose that no all of these multipliers turn out to be positive. Then, the guess that the case under consideration is an optimum is wrong. How do we proceed then? The KKT conditions provide us with the information of how much we are willing to pay for an extra unit of portfolio component at each constraint: A strictly positive value if a constraint is truly binding, zero if the constraint is not binding and a negative value if we assumed that a component will bind the case under consideration but in effect it does not. To proceed analytically using this information is very cumbersome if the number of assets is larger than 2. But fortunately there exist powerful algorithms which can do this routine checking systematically and for a large class of assets (see for example the *Active Direction Algorithm*). But to learn about the impact of restrictions, it is nevertheless unavoidable to work analytically. Before we consider examples, we analyze the general properties of the feasible set A if linear investment restrictions are imposed.

We formulate the restricted optimization problem in the form

$$\begin{aligned}\sigma^2_{rest}(z) &= \min_{\phi}\{\langle\phi, V\phi\rangle \mid \langle e,\phi\rangle = 1\ ,\ \langle\mu,\phi\rangle = r\ ,\ -\phi \geq -b\ ,\ \phi \geq a\} \\ &= \min_{\phi}\{\langle\phi, V\phi\rangle \mid G(\phi) \geq z_{rest}\}\end{aligned}$$

with $z_{rest} = (1, r, a, -b)'$ and $G(\phi) = (\langle e,\phi\rangle, \langle\mu,\phi\rangle, \phi, -\phi)'$. The function $\sigma^2_{rest}(z_{res}$ is convex and increasing if $r \geq r_*$ (see Chapter **??**, Proposition **??**). We define the following vectors:

$$\begin{aligned} z &= (1, r, 0, -1)\ ,\ \text{(unrestricted investment)} \\ z_{down} &= (1, r, a, -1)\ ,\ \text{(downside restrictions)} \\ z_{up} &= (1, r, 0, -b)\ ,\ \text{(upside restrictions)} \end{aligned}$$

and we denote with A the feasible set in the unrestricted case, with A_{down} in the downside case and with A_{up} in the upside case, respectively. For a given expected return r, we have

$$z \leq z_{down} \leq z_{rest}\ ,\ z \leq z_{up} \leq z_{rest}$$

which implies

$$\sigma^2(z) \leq \sigma^2(z_{down}) \leq \sigma^2(z_{rest})\ ,\ \sigma^2(z) \leq \sigma^2(z_{up}) \leq \sigma^2(z_{rest})\ .$$

Therefore, in the case of restrictions it is not possible to attain the same expected level of return r as in the unrestricted case with lower risk. This implies for the feassible sets

$$A_{rest} \subset A_{down} \subset A\ ,\ A_{rest} \subset A_{up} \subset A\ .$$

We note, that there exists no ordering between z_{down} and z_{up}.

The simplest **example** we consider next is the two asset case with a single restriction for one asset, i.e.

$$\min_{\phi} \quad \frac{1}{2}\langle\phi, V\phi\rangle \qquad (\mathcal{M}_{rest,example}) \tag{54}$$

$$s.t. \quad S_{rest} = \{\phi \in \mathbf{R}^2 \mid \langle e,\phi\rangle = 1\ ,\ \langle\mu,\phi\rangle = r\ ,\ a_1 \leq \phi_1\ ,\ 0 \leq a_1\}\ . \tag{55}$$

The optimality conditions then read

$$0 = V\phi - \lambda_1 e - \lambda_2\mu - Da_1\ ,\ D = \begin{pmatrix}\lambda_3 & 0\\ 0 & 0\end{pmatrix} \tag{56}$$

$$\begin{aligned} &\langle e,\phi\rangle = 1\ ,\ \langle\mu,\phi\rangle = r \\ &\lambda_3 \geq 0 \\ &\lambda_3(a_1 - \phi_1) = 0\ . \end{aligned} \tag{57}$$

To solve the optimality conditions for the optimal investment and the optimal multipliers different case have to be distinguished.

Case I

We assume that $\phi_1 > a_1$, i.e. that the restriction on investment in the first asset is not binding. The complementary slackness condition $\lambda_3(a_1 - \phi_1) = 0$ then implies that $ll_3 = 0$ which (i) is a positive value and (ii) clearly leads to the optimal solution ϕ^* of the unresticted problem given in Proposition .

Case II

In the second case, which is the only one missing in this example, we assume that $\phi_1 = a_1$. Hence we get for the **three** multipliers the following linear equation system to solve:

$$\begin{aligned}
\langle e, \phi\rangle &= 1 = \lambda_1 b + \lambda_2 c + \lambda_3 a_1 && (58)\\
\langle \mu, \phi\rangle &= r = \lambda_1 c + \lambda_2 a + \lambda_s \mu_1 a1\\
\phi_1 &= a_1 = \lambda_1 (V^{-1}e)_1 + \lambda_2 (V^{-1}\mu)_1 + \lambda_3 a_1 && (59)
\end{aligned}$$

with $(x)_i$ the ith component of the vector x. Solving the system implies

$$\begin{aligned}
\lambda_1^{rest} &= \frac{a_1(c\mu_1 - a) + a - rc + (V^{-1}\mu)_1(r - \mu_1)}{-\Delta + (V^{-1}e)_1(a - \mu_1 c) + (V^{-1}\mu)_1(\mu_1 b - c)} && (60)\\
\lambda_2^{rest} &= \frac{a_1(c - b\mu_1) + br - c + (V^{-1}e)_1(\mu_1 - r)}{-\Delta + (V^{-1}e)_1(a - \mu_1 c) + (V^{-1}\mu)_1(\mu_1 b - c)} && (61)\\
\lambda_3^{rest} &= \frac{a_1\Delta + (V^{-1}e)_1(a - rc) + (V^{-1}\mu)_1(br - c)}{-a_1\left(-\Delta + (V^{-1}e)_1(a - \mu_1 c) + (V^{-1}\mu)_1(\mu_1 b - c)\right)} && (62)\\
& && (63)
\end{aligned}$$

If we recall the optimal multipliers for the unrestricted problem

$$\begin{aligned}
\lambda_1 &= \frac{1}{\Delta}(-a + rc)\\
\lambda_2 &= \frac{1}{\Delta}(-c + rb)
\end{aligned}$$

it follows, that the first two multipliers of the restricted problem are equal to the unrestricted multipliers plus a correction. Although the expressions complicated, they are still affine functions of r. If we insert the optimal multipliers in the optimality condition for ϕ_2, the optimal investment in the second asset is found. From the structure of the multipliers and the affine form of ϕ_2^* follows, that the efficient frontier is still a hyperbola for the investments (a_1, ϕ_2^*). But the form of the hyperbola is different than that one for the investment $(\phi_1^* > a_1, \phi_2^*)$.

In the next **example** we go a step further, i.e. we consider still two risky assets but both of them are restricted below and above. If we restrict to the two asset case the problem is still manageable. Let

$$D_1 = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix}, \; D_2 = \begin{pmatrix} \lambda_5 & 0 \\ 0 & \lambda_6 \end{pmatrix}$$

and the optimality conditions are with the Assumptions 6

$$\begin{aligned}
y &= A\tau + w && (64)\\
& \lambda_j \geq 0 \;, j = 3,4,5,6\\
& \lambda_j(a_j - \phi_j) = 0 \;,\; j = 3,4 \;,\; \lambda_j(\phi_j - b_j) = 0 \;,\; j = 5,6
\end{aligned}$$

with

$$w = \begin{pmatrix} \langle e, V^{-1}D_1 a\rangle - \langle e, V^{-1}D_2 a\rangle \\ \langle \mu, V^{-1}D_1 b\rangle - \langle \mu, V^{-1}D_2 b\rangle \end{pmatrix} \qquad (65)$$

and y, A, τ defined in (9). Since A is invertible, we get for the multipliers $\tau = (\lambda_1, \lambda_2)'$

$$\tau = A^{-1}y - A^{-1}w \tag{66}$$

which are functions (due to the vector w) of the other multipliers $\lambda_3, \ldots, \lambda_6$. If the solution turns out to be an interior one, $\lambda_3, \ldots, \lambda_6$ all are zero and τ and as well ϕ^* are the same as for the unrestricted problem.

To proceed we consider a single case out of the $3^2 = 9$ possible cases. We assume that the first component of the optimal portfolio binds ($\phi_1^* = a_1$) and that the second one is an interior solution ($a_2 < \phi_2^* < b_2$). From the normalization condition we get $a_2 = 1 - a_1$ and we assume $a_2 < 1 - a_1 < b_2$. Therefore the multipliers $\lambda_4 = \lambda_6 = 0$ for the second component and also $\lambda_5 = 0$ since the first component can not bind at the lower boundary *and* at the higher boundary simultaneously. We furthermore assume that the constraints are *generic*, i.e. that the consistency conditions (**??**) are satisfied and that any summation over any two constraints does not equal 1.

For the case under consideration, we get

$$w = \lambda_3 a_1 \begin{pmatrix} V_{11}^{-1} + V_{21}^{-1} \\ \mu_1 V_{11}^{-1} + \mu_2 V_{21}^{-1} \end{pmatrix} =: \lambda_3 a_1 s \ . \tag{67}$$

Hence,

$$\lambda_i = (A^{-1}y)_i - \lambda_3 a_1 (A^{-1}y)_i \ , \ i = 1, 2 \ . \tag{68}$$

This implies that once λ_3 is determined and and $\lambda_3 \geq 0$ verified, λ_1, λ_2 can be determined from (68). The portfolio $\phi = (a_1, 1-a_1)'$ is then, by the KKT theorem, the solution of the restricted two-asset problem. To calculate λ_3 we use

$$\begin{aligned} \phi_1^* &= a_1 = \lambda_1 (V^{-1}e)_1 + \lambda_2 (V^{-1}\mu)_1 + (V^{-1}D_1 a)_1 \\ &= \left((A^{-1}y)_1 - \lambda_3 a_1 (A^{-1}y)_1\right)(V^{-1}e)_1 + \left((A^{-1}y)_2 - \lambda_3 a_1 (A^{-1}y)_2\right)(V^{-1}\mu)_1 \\ &+ \lambda_3 a_1 V_{11}^{-1} \ . \end{aligned}$$

Solving for λ_3 and defining the vectors

$$m = ((A^{-1}y)_1, (A^{-1}y)_2)' \ , \ n = ((V^{-1}e)_1, (V^{-1}\mu)_1)' \tag{69}$$

we get

$$\lambda_3 = \frac{a_1 - \langle m, n \rangle}{a_1 (V_{11}^{-1} - \langle m, n \rangle)} \ . \tag{70}$$

So far, we can state:

Whether the optimal portfolio binds at a restriction and satisfies the KKT condition that the respective multiplier is positive can not be answered in a straightforward way. This is due to the fact that all parameters enter in the conditions in a non-trivial way.

Since $\det V, a_1 > 0$, also $V_{11}^{-1} = \frac{1}{\det V} V_{22}$ is positive, and a sufficient condition for $\lambda_3 > 0$ in (70) is

$$\langle m, n \rangle < 0 \ . \tag{71}$$

In detail, this sufficiency condition reads

$$\begin{aligned} \langle m, n \rangle &= (a - rc)(V_{22} - V_{12}) \\ &+ (rb - b)(\mu_1 V_{22} - \mu_2 V_{12}) < 0 \ . \end{aligned} \tag{72}$$

To analyze further when λ_3 is positive we plot this multiplier as a function of the returns μ_1, μ_2 for different model parameters.

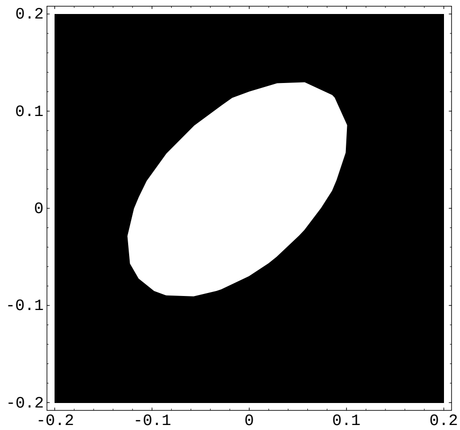

FIGURE 10. The benchmark example. The expected return is 4 percent ($r = 0.04$) and the covariance matrix V is $V = \begin{pmatrix} 0.4 & 0.18 \\ 0.18 & 0.37 \end{pmatrix}$. The two assets are therefore positively correlated. The figure shows a level plot, i.e. $\lambda_3(\mu_1, \mu_2) =$ constant, where the critical level $\lambda_3(\mu_1, \mu_2) = 0$ is shown as the frontier curve between the white and the black region. In the white region λ_3 is positive. On the horizontal axis μ_1 varies from -20 percent to $+20$ percent and on the vertical axis μ_2 varies in the same range. The restriction a_1 is equal to 0.006.

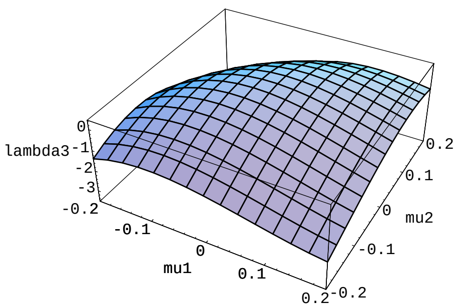

FIGURE 11. A 3-dimensional plot of the benchmark case.

The numerical values of the benchmark example are given in Figure **??** and the variations of the benchmark are in the subsequent figures. In the benchmark example, it follows that for varying returns of the two assets λ_3 is positive in a bounded domain for the returns of *both* assets (see Figure **??**). Therefore, if either of the returns is very high or low, the case that asset 1 binds at a_1 can not be optimal. Therefore, for a fixed covariance matrix there exist large enough returns (positive and negative one) relative to the fixed risk structure such that binding at the a_1 is never optimal. This boundness is surprising since one would expect that for very negative returns a binding of the portfolio at the lower constraint. But it

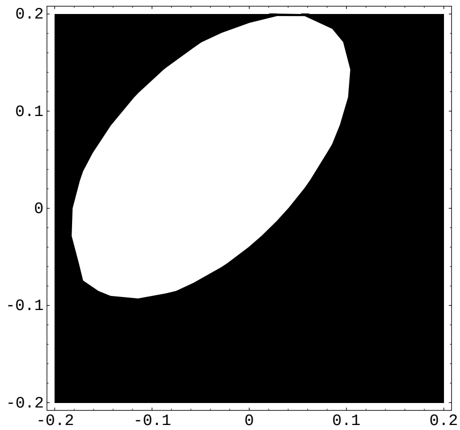

FIGURE 12. In this variation of the benchmark example the expected return $r = 0.04$ is raised to 12 percent. All other parameter are unchanged. In the white region λ_3 is positive.

follows from the figure, that a possible binding for the asset 1 at the lower constraint is only asymmetric with respect to the sign of the return μ_1. The return μ_1 may assume more negative values than positive ones such that the binding portfolio is feasible. A further property is the position of the main axis of the ellipse-like curve $\lambda_3(\mu_1, \mu_2) = 0$. It follows that for a given small enough positive return μ_1, a positive return μ_2 always exists such that asset 1 binds. Contrary negative returns μ_2 are more unlikely to exist such that $\lambda_3 \geq 0$ holds true; but nevertheless they can exist.
In summary, the benchmark example shows that the following naive intuitions are wrong:

(1) Whether or not assets bind is only due to their return properties and their relative comparison. In fact all parameters of the model matter.
(2) If returns become arbitrary negative, the assets will bind at the lower constraints.

In the benchmark, a_1 is chosen to be equal to the low value 0.006. If we increase a_1 up to 30 percent, then $\lambda_3 > 0$ in the whole parameter domain of the benchmark example. Therefore, if the lower restriction is set large enough and if the optimum binds, then the investor possesses a willingness to pay for a lower value of the constraint. This is equivalent to $\lambda_3 > 0$.
In Figure 4) the expected return r was altered from 4 to 12 percent. Then the white region, which represents $\lambda_3 \geq 0$, is parallel shifted in direction of the second quadrant. Consider a pair of positive returns μ_1, μ_2, μ_2 smaller thanμ_1, which in the benchmark model led to $\lambda_3 > 0$. If the expected return is raised, then the constraint will typically no longer bind. But this makes sense, since for the investor has to give up the binding constraint for asset 1, which has by assumption a larger return than asset 2, else he can not cope with the raised expected return.
In Figure 8) the asset are negatively correlated. This has two effects: First the region where $\lambda_3 \geq 0$ in the benchmark is rotated by 90 degrees and second, the white domain, representing $\lambda_3 > 0$ has a larger size. The second effect is clear, since the two assets are "substitutes" in this case and no longer "complements".

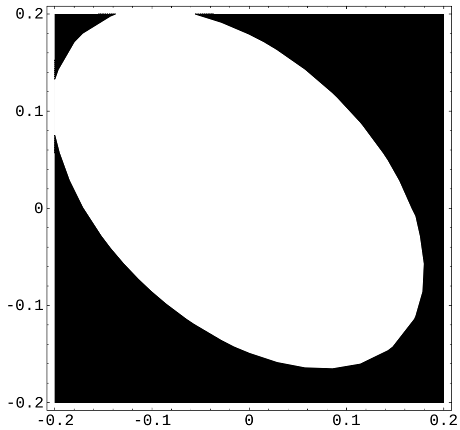

FIGURE 13. In this variation of the benchmark example, we assumed that the assets are negatively correlated. The covariance matrix is $V = \begin{pmatrix} 0.4 & -0.18 \\ -0.18 & 0.37 \end{pmatrix}$. In the white region λ_3 is positive.

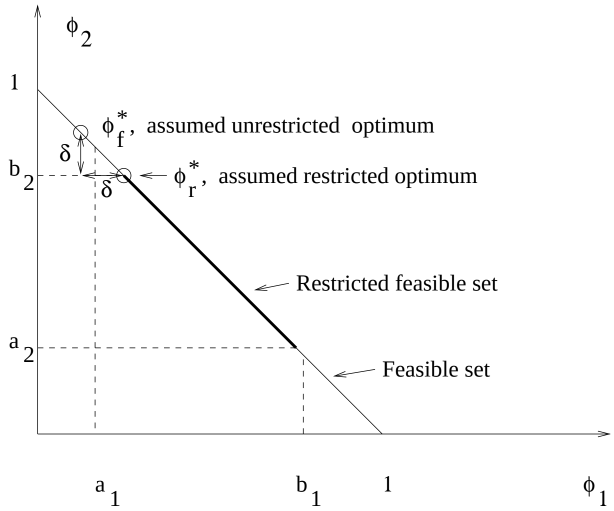

FIGURE 14. The impact of restrictions on efficiency.

The terminology is defined as follows. Let $j \neq k$ and consider

$$\frac{\partial^2 W(\phi)}{\partial \phi_j \partial \phi_k} = \begin{cases} > 0 & \Longleftrightarrow \ \phi_j \text{ and } \phi_k \text{ are complements} \\ < 0 & \Longleftrightarrow \ \phi_j \text{ and } \phi_k \text{ are substitutes} \end{cases} .$$

where W is the value or utility function of the good vector ϕ. In our case $W = \langle \phi, V^{\phi} \rangle$ and the "goods" is the portfolio. Therefore, if in the mean-variance model a unit more invested in asset j decreases the marginal value of asset k, then we substitute k with assets j.

Given this numerical examples for two assets only, we can state for the general case of $N > 2$ assets, that it is mostly impossible to decide without using a computer to determine which restrictions in the model bind.

We next consider the impact of restrictions on efficiency (see Figure 8 for the setup). The restricted expected portfolio return $E[R^{\phi_r^*}]$ is given by

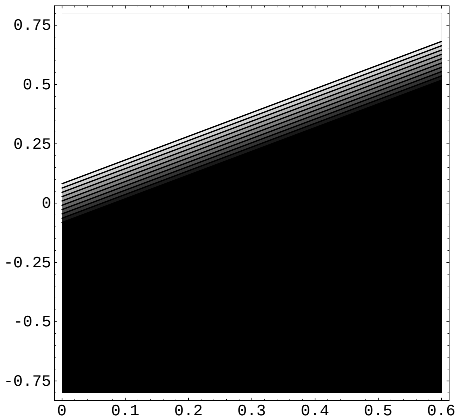

FIGURE 15. Case B: The level-plot illustrates the signs of H in the benchmark example defined as follows: We assume that the unrestricted optimal portfolio is $\phi_f^* = (\phi_1^*, \phi_2^*) = (0.2, 0.8)$, that $V_{11} = V_{22}$ and that the correction $\delta = 0.1$. On the horizontal axis V_{22} varies and on the vertical axis the correlation V_{12}. We note that in the black region the variance of the restricted problem is **larger** than the variance of the unrestricted problem.

$$E[R^{\phi_r^*}] = \mu_1(\phi_1^* + \delta) + \mu_2(\phi_2^* - \delta) = E[R^{\phi_f^*}] + \delta(\mu_1 - \mu_2) \quad (73)$$

with $\phi_i^*, i = 1, 2$, the components of the unrestricted optimal portfolio ϕ_f^*. For the respective variance of the portfolios we have

$$var(R^{\phi_r^*}) = var(R^{\phi_f^*}) + var(R^d) + 2\langle d, V\phi_f^* \rangle \quad (74)$$

where

$$d = (\delta, -\delta)' \ , \ var(R^d) = \langle d, Vd \rangle \ . \quad (75)$$

Since the fraction of wealth invested in asset 2 is larger than in asset 1 in the unrestricted case, we assume that $\mu_2 > \mu_1$. Therefore, the constraint implies in this case

$$E[R^{\phi_r^*}] = r_{rest}^* < E[R^{\phi_f^*}] = r^* \ ,$$

i.e. the expected return is smaller for the restricted, optimal portfolio than in the unrestrictd case. We analyze when

$$var(R^{\phi_r^*}) > var(R^{\phi_f^*})$$

holds, which then implies that the constraints enforce to choose a dominated portfolio.

The condition $var(R^{\phi_r^*}) > var(R^{\phi_f^*})$ is equivalent to (assuming that $V_{12} = V_{21}$)

$$H = \delta^2(V_{11} + V_{22} - 2V_{12}) + 2\delta(\phi_1 V_{11} - \phi_2 V_{22} + V_{12}(\phi_2 - \phi_1)) > 0 \ . \quad (76)$$

We consider the sign of H in the examples shown in Figures 8 to 8. The values of the expression H are shown as a function of the variance V_{22} and the correlation V_{12} between the assets.

It follows that a variation in the relative magnitudes between the variances of the two assets has a very weak impact on the the sign of the function H.

We discuss the content of the figures 8 (benchmark case B), 8 (higher unrestricted value ϕ_2^*, case 2) and 8 (larger correction δ, case 3). It follows, given the model parameters, the possibility that a dominated portfolio is selected is higher (relative to the benchmark) if the correction δ is larger. Therefore, the more severe are the

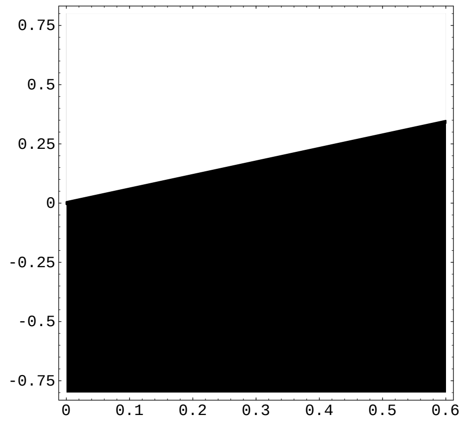

FIGURE 16. Case 2: The same parameters hold as in the benchmark example but the optimal unrestricted portfolio component is assumed to be $\phi_2^* = 0.9$.

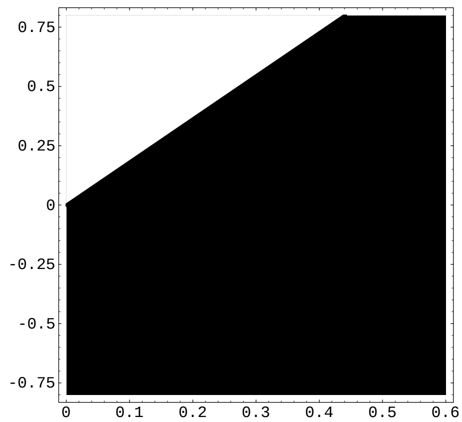

FIGURE 17. Case 3: The same parameters hold as in the benchmark example but the correction of portfolio fractions is augmented to $\delta = 0.3$ from $\delta = 0.1$.

restrictions acting on the unrestricted optimum, the more will dominated portfolios be selected. Contrary if the unrestricted, optimal value ϕ_2^2 is larger than in the benchmark case, the possibility of the restrictions leading to dominated portfolios is lower than in the benchmark case.
For the further discussion, consider Figure 8. The figure indicates graphically the possible location ordering of the three cases under consideration. It also shows the impact of changing factors in the correlation matrix. If the two assets are positively correlated, a restriction on the "better" asset ϕ_2 leads for an increasing correlation to a non-dominated portfolio o. This is intuitively clear since substitution takes place with an asset which is "in line" with the substituted asset. Contrary, if substitution is with a strongly negatively correlated asset, the possibility of a dominated portfolio choice increases.

In summary, the following insights are gained in this section.

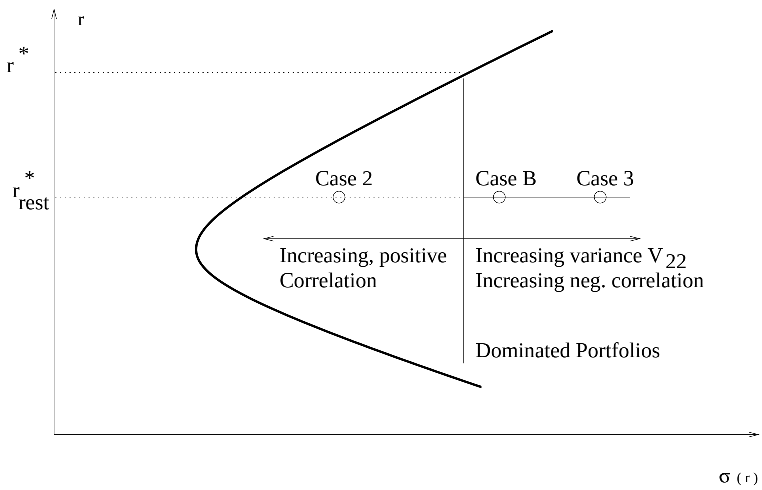

FIGURE 18. Comparison of the cases B, 2 and 3, respectively, in the $(\sigma(r), r)$-space.

- Mean-variance analysis with linear investment restriction is simple in principle to carry out but already for a few number of assets, there is a large number of cases to be analyzed.
- Referring to the next section, there are powerful numerical methods to analyze portfolio selection problem with linear investment restrictions.
- Whether or not assets bind at their restriction values is not only due to their return properties, but all parameters of the model matter in a complicated way.
- Restrictions may well lead to dominated portfolio choices relative to the unrestricted selection. Again, this possibility depends on all parameter of the model in an essentially non-trivial way.

If we consider these facts, we conclude that we have to separate restrictions which (i) are due to market rules and (ii) restrictions imposed by regulatory authorities or by the asset managers themselves.
While the first type of rules have to be incorporated into portfolio selection models, such as existing short sale restrictions for example, restrictions of the type (ii) should be avoided. To understand the reason for this statement consider we consider the situation where fund managers are interested in an optimal asset allocation. Suppose they agree to use mean and variance as their decision criteria. Typically, asset managers then like to impose asset restrictions of the linear type discussed in this section and in a second step, the restricted mean-variance problem is solved. The reason to choose restrictions first, is due to their intuition that the result has to be "well-diversified", i.e. "The fraction invested in asset j should not be smaller and higher than the prescribed values x and y respectively". But this contradicts the logic of the model. Since the unrestricted model provides the managers already with the optimal diversification, there is no need to control this solution. It could also be, that the fund managers "know" what is reasonable and they want to prevent that a "unreasonable" solution occurs. In any case, this behavior reflects that the mean-variance model under consideration is not appropriate for decision making and a sound modelling should be considered in such situations. In other words, another objective function should be chosen and not the admissible set restricted. In the following chapters we will see what happens if other objectives are agreed upon than the mean and variance of a portfolio.

9. Equivalent formulations of mean-variance portfolio selection

In the discussion of the mean-variance selection problem we always worked with the model $\mathcal{M}$. In fact, there exist other model formulations which are equivalent to this model. This means, that they lead to an optimal portfolio choice with the same expected return and variance than in the case considered so far.
A possible model is

$$\min_{\psi} \quad \frac{1}{2}\langle\psi, V\psi\rangle - \rho\langle\mu, \phi\rangle \quad ,(\mathcal{M}_1) \tag{77}$$

$$s.t. \quad \langle\psi, e\rangle = 1, \psi \in \mathbf{R}^N, \tag{78}$$

with ρ an exogenous parameter. The solution of this model is

$$\psi^* = V^{-1}(\rho\mu + \lambda e)\ ,\ \lambda = \frac{1-\rho c}{b}\ . \tag{79}$$

The two portfolios ψ^* and ϕ^* of Proposition 8 are equivalent, iff

$$\langle\psi^*, \mu\rangle = \langle\phi^*, \mu\rangle\ ,\ \sigma^2_{\phi^*}(r) = \sigma^2_{\psi^*}(\rho)\ , \tag{80}$$

where in the risk measure the explicit dependence on the respective model parameters are shown. A short calculation shows that the variance condition is always satisfied iff

$$r = \frac{\rho\Delta + c}{b}\ . \tag{81}$$

To prove this, we calculate

$$\langle\psi^*, V\psi^*\rangle = \rho^2\frac{\Delta}{b} + \frac{1}{b}$$

and compare it with

$$\langle\phi^*, V\phi^*\rangle = \frac{1}{\Delta}(r^2 b - 2rc + a)\ .$$

The linear relation between r and ρ in (81) guarantees that the expected returns and the variances in both models agree. Therefore, for any choice of desired expected return r there exist a value ρ such that the model $\mathcal{M}_1$ leads to the same outcome. This proves that the two models are equivalent.
Although the two models are mathematically equivalent they may *not* be equivalent from a behavioral point of view. Consider an investor which plans to find out its optimal one-period investment according to the mean-variance criterion. In the model $\mathcal{M}$ he needs then to fix the desired rate r, whereas in the model $\mathcal{M}_1$ he has to determine the trade-off between risk and return by selecting the parameter ρ. The equivalence of the models designs an experiment for a laboratory to test peoples rationality in decision making under uncertainty.